C.4 Applications of Differential Equations

Use differential equations to model and solve real-life problems.

EXAMPLE 1 Modeling Advertising Awareness

The new cereal product from Example 3 in Section C.1 is introduced through an advertising campaign to a population of 1 million potential customers. The rate at which the population hears about the product is assumed to be proportional to the number of people who are not yet aware of the product. By the end of 1 year, half of the population has heard of the product. How many will have heard of it by the end of 2 years?

Solution Let y be the number (in millions) of people at time t who have heard of the product. This means that $(1 - y)$ is the number of people who have not heard, and dy/dt is the rate at which the population hears about the product. From the given assumption, you can write the differential equation as follows.

$$\underbrace{\frac{dy}{dt}}_{\text{Rate of change of } y} \; \underbrace{=k}_{\text{is proportional to}} \; \underbrace{(1-y)}_{\text{the difference between 1 and } y}$$

Using separation of variables *or* a symbolic integration utility, you can find the general solution to be

$$y = 1 - Ce^{-kt}. \qquad \text{General solution}$$

To solve for the constants C and k, use the initial conditions. That is, because $y = 0$ when $t = 0$, you can determine that $C = 1$. Similarly, because $y = 0.5$ when $t = 1$, it follows that $0.5 = 1 - e^{-k}$, which implies that $k = \ln 2 \approx 0.693$. So, the particular solution is

$$y = 1 - e^{-0.693t}. \qquad \text{Particular solution}$$

This model is shown graphically in Figure A.13. Using the model, you can determine that the number of people who have heard of the product after 2 years is

$$\begin{aligned} y &= 1 - e^{-0.693(2)} \\ &\approx 0.75 \text{ or } 750{,}000 \text{ people.} \end{aligned}$$

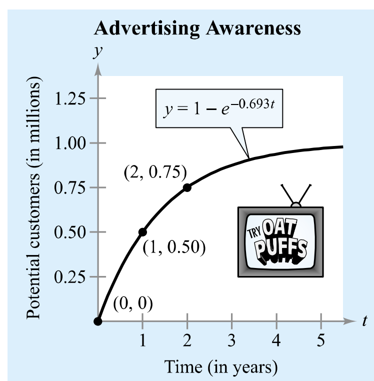


FIGURE A.13

EXAMPLE 2 Modeling a Chemical Reaction

During a chemical reaction, substance A is converted into substance B at a rate that is proportional to the square of the amount of A. When $t = 0$, 60 grams of A are present, and after 1 hour ($t = 1$), only 10 grams of A remain unconverted. How much of A is present after 2 hours?

Solution Let y be the unconverted amount of substance A at any time t. From the given assumption about the conversion rate, you can write the differential equation as follows.

$$\frac{dy}{dt} = ky^2$$

Rate of change of y — is proportional to — the square of y.

Using separation of variables *or* a symbolic integration utility, you can find the general solution to be

$$y = \frac{-1}{kt + C}. \qquad \text{General solution}$$

To solve for the constants C and k, use the initial conditions. That is, because $y = 60$ when $t = 0$, you can determine that $C = -\frac{1}{60}$. Similarly, because $y = 10$ when $t = 1$, it follows that

$$10 = \frac{-1}{k - (1/60)}$$

which implies that $k = -\frac{1}{12}$. So, the particular solution is

$$y = \frac{-1}{(-1/12)t - (1/60)} \qquad \text{Substitute for } k \text{ and } C.$$

$$= \frac{60}{5t + 1}. \qquad \text{Particular solution}$$

Using the model, you can determine that the unconverted amount of substance A after 2 hours is

$$y = \frac{60}{5(2) + 1}$$
$$\approx 5.45 \text{ grams.}$$

In Figure A.14, note that the chemical conversion is occurring rapidly during the first hour. Then, as more and more of substance A is converted, the conversion rate slows down.

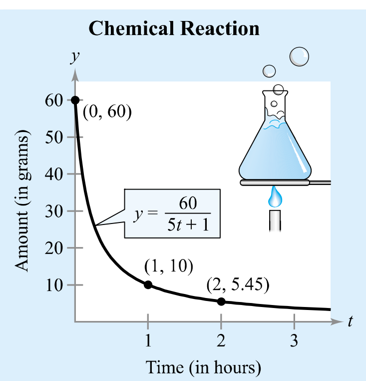


FIGURE A.14

STUDY TIP

In Example 2, the rate of conversion was assumed to be proportional to the square of the unconverted amount. How would the result change if the rate of conversion were assumed to be proportional to the unconverted amount?

Earlier in the text, you studied two models for population growth: *exponential growth*, which assumes that the rate of change of y is proportional to y, and *logistic growth*, which assumes that the rate of change of y is proportional to y and $L - y$, where L is the population limit.

The next example describes a third type of growth model called a **Gompertz growth model.** This model assumes that the rate of change of y is proportional to y and the natural log of L/y, where L is the population limit.

EXAMPLE 3 Modeling Population Growth

A population of 20 wolves has been introduced into a national park. The forest service estimates that the maximum population the park can sustain is 200 wolves. After 3 years, the population is estimated to be 40 wolves. If the population follows a Gompertz growth model, how many wolves will there be 10 years after their introduction?

During the second half of the twentieth century, wolves disappeared from most of the middle and northern areas of the United States. Recently, however, wolf populations have been reappearing in several northern national parks.

Solution Let y be the number of wolves at any time t. From the given assumption about the rate of growth of the population, you can write the differential equation as follows.

$$\frac{dy}{dt} = ky \ln \frac{200}{y}$$

Rate of change of y | is proportional to | the product of y and | the log of the ratio of 200 and y.

Using separation of variables *or* a symbolic integration utility, you can find the general solution to be

$$y = 200e^{-Ce^{-kt}}. \qquad \text{General solution}$$

To solve for the constants C and k, use the initial conditions. That is, because $y = 20$ when $t = 0$, you can determine that

$$C = \ln 10 \approx 2.3026.$$

Similarly, because $y = 40$ when $t = 3$, it follows that

$$40 = 200e^{-2.3026e^{-3k}}$$

which implies that $k \approx 0.1194$. So, the particular solution is

$$y = 200e^{-2.3026e^{-0.1194t}}. \qquad \text{Particular solution}$$

Using the model, you can estimate the wolf population after 10 years to be

$$y = 200e^{-2.3026e^{-0.1194(10)}}$$
$$\approx 100 \text{ wolves.}$$

In Figure A.15, note that after 10 years the population has reached about half of the estimated maximum population. Try checking the growth model to see that it yields $y = 20$ when $t = 0$ and $y = 40$ when $t = 3$.

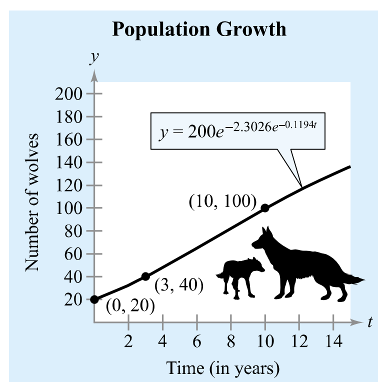


FIGURE A.15

In genetics, a commonly used hybrid selection model is based on the differential equation

$$\frac{dy}{dt} = ky(1 - y)(a - by).$$

In this model, y represents the portion of the population that has a certain characteristic and t represents the time (measured in generations). The numbers a, b, and k are constants that depend on the genetic characteristic that is being studied.

EXAMPLE 4 Modeling Hybrid Selection

You are studying a population of beetles to determine how quickly characteristic D will pass from one generation to the next. At the beginning of your study ($t = 0$), you find that half the population has characteristic D. After four generations ($t = 4$), you find that 80% of the population has characteristic D. Use the hybrid selection model above with $a = 2$ and $b = 1$ to find the percent of the population that will have characteristic D in 10 generations.

Solution Using $a = 2$ and $b = 1$, the differential equation for the hybrid selection model is

$$\frac{dy}{dt} = ky(1 - y)(2 - y).$$

Using separation of variables *or* a symbolic integration utility, you can find the general solution to be

$$\frac{y(2 - y)}{(1 - y)^2} = Ce^{2kt}. \qquad \text{General solution}$$

To solve for the constants C and k, use the initial conditions. That is, because $y = 0.5$ when $t = 0$, you can determine that $C = 3$. Similarly, because $y = 0.8$ when $t = 4$, it follows that

$$\frac{0.8(1.2)}{(0.2)^2} = 3e^{8k}$$

which implies that

$$k = \frac{1}{8}\ln 8 \approx 0.2599.$$

So, the particular solution is

$$\frac{y(2 - y)}{(1 - y)^2} = 3e^{0.5199t}. \qquad \text{Particular solution}$$

Using the model, you can estimate the percent of the population that will have characteristic D in 10 generations to be given by

$$\frac{y(2 - y)}{(1 - y)^2} = 3e^{0.5199(10)}.$$

Using a symbolic algebra utility, you can solve this equation for y to obtain $y \approx 0.96$. The graph of the model is shown in Figure A.16.

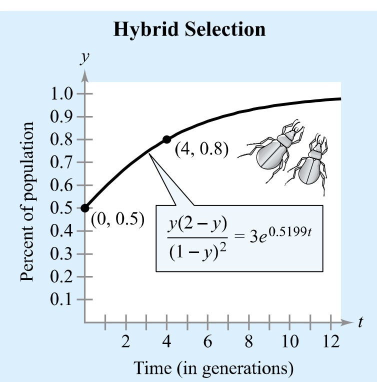


FIGURE A.16

EXAMPLE 5 Modeling a Chemical Mixture

A tank contains 40 gallons of a solution composed of 90% water and 10% alcohol. A second solution containing half water and half alcohol is added to the tank at the rate of 4 gallons per minute. At the same time, the tank is being drained at the rate of 4 gallons per minute, as shown in Figure A.17. Assuming that the solution is stirred constantly, how much alcohol will be in the tank after 10 minutes?

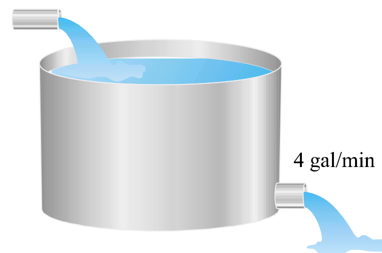


FIGURE A.17

Solution Let y be the number of gallons of alcohol in the tank at any time t. The *percent* of alcohol in the 40-gallon tank at any time is $y/40$. Moreover, because 4 gallons of solution are being drained each minute, the rate of change of y is

$$\underbrace{\frac{dy}{dt}}_{\text{Rate of change of } y} = \underbrace{-4\left(\frac{y}{40}\right)}_{\text{is equal to the amount of alcohol draining out}} + \underbrace{2}_{\text{plus the amount of alcohol entering.}}$$

where 2 represents the number of gallons of alcohol entering each minute in the 50% solution. In standard form, this linear differential equation is

$$y' + \frac{1}{10}y = 2. \qquad \text{Standard form}$$

Using an integrating factor *or* a symbolic integration utility, you can find the general solution to be

$$y = 20 + Ce^{-t/10}. \qquad \text{General solution}$$

Because $y = 4$ when $t = 0$, you can conclude that $C = -16$. So, the particular solution is

$$y = 20 - 16e^{-t/10}. \qquad \text{Particular solution}$$

Using this model, you can determine that the amount of alcohol in the tank when $t = 10$ is

$$y = 20 - 16e^{-(10)/10}$$
$$\approx 14.1 \text{ gallons.}$$

TAKE ANOTHER LOOK

Chemical Mixture

Sketch the particular solution obtained in Example 5. Describe the rate of change of the amount of alcohol in the tank. Does the amount approach 0 as t increases? Explain your reasoning.

WARM-UP C.4

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section.

In Exercises 1–4, use separation of variables to find the general solution of the differential equation.

1. $\dfrac{dy}{dx} = 3x$

2. $2y\dfrac{dy}{dx} = 3$

3. $\dfrac{dy}{dx} = 2xy$

4. $\dfrac{dy}{dx} = \dfrac{x - 4}{4y^3}$

In Exercises 5–8, use an integrating factor to solve the first-order linear differential equation.

5. $y' + 2y = 4$

6. $y' + 2y = e^{-2x}$

7. $y' + xy = x$

8. $xy' + 2y = x^2$

In Exercises 9 and 10, write the equation that models the statement.

9. The rate of change of y with respect to x is proportional to the square of x.

10. The rate of change of x with respect to t is proportional to the difference of x and t.

EXERCISES C.4

In Exercises 1–6, assume that the rate of change of y is proportional to y. Solve the resulting differential equation $dy/dx = ky$ and find the particular solution that passes through the points.

1. $(0, 1), (3, 2)$

2. $(0, 4), (1, 6)$

3. $(0, 4), (4, 1)$

4. $(0, 60), (5, 30)$

5. $(2, 2), (3, 4)$

6. $(1, 4), (2, 1)$

7. ***Investment*** The rate of growth of an investment is proportional to the amount A of the investment at any time t. An investment of \$2000 increases to a value of \$2983.65 in 5 years. Find its value after 10 years.

8. ***Population Growth*** The rate of change of the population of a city is proportional to the population P at any time t. In 1998, the population was 400,000, and the constant of proportionality was 0.015. Estimate the population of the city in the year 2005.

9. ***Sales Growth*** The rate of change in sales S (in thousands of units) of a new product is proportional to the difference between L and S (in thousands of units) at any time t. When $t = 0$, $S = 0$. Write and solve the differential equation for this sales model.

10. ***Sales Growth*** Use the result of Exercise 9 to write S as a function of t if (a) $L = 100$, $S = 25$ when $t = 2$, and (b) $L = 500$, $S = 50$ when $t = 1$.

In Exercises 11–14, the rate of change of y is proportional to the product of y and the difference of L and y. Solve the resulting differential equation $dy/dx = ky(L - y)$ and find the particular solution that passes through the points for the indicated value of L.

11. $L = 20$; $(0, 1), (5, 10)$

12. $L = 100$; $(0, 10), (5, 30)$

13. $L = 5000$; $(0, 250), (25, 2000)$

14. $L = 1000$; $(0, 100), (4, 750)$

15. ***Biology*** At any time t, the rate of growth of the population N of deer in a state park is proportional to the product of N and $L - N$, where $L = 500$ is the maximum number of deer the park can maintain. When $t = 0$, $N = 100$, and when $t = 4$, $N = 200$. Write N as a function of t.

16. ***Sales Growth*** The rate of change in sales S (in thousands of units) of a new product is proportional to the product of S and $L - S$. L (in thousands of units) is the estimated maximum level of sales, and $S = 10$ when $t = 0$. Write and solve the differential equation for this sales model.

Learning Theory In Exercises 17 and 18, assume that the rate of change in the proportion P of correct responses after n trials is proportional to the product of P and $L - P$, where L is the limiting proportion of correct responses.

17. Write and solve the differential equation for this learning theory model.

18. Use the solution of Exercise 17 to write P as a function of n, and then use a graphing utility to graph the solution.

(a) $L = 1.00$

$P = 0.50$ when $n = 0$

$P = 0.85$ when $n = 4$

(b) $L = 0.80$

$P = 0.25$ when $n = 0$

$P = 0.60$ when $n = 10$

Chemical Reaction In Exercises 19 and 20, use the chemical reaction model in Example 2 to find the amount y as a function of t, and use a graphing utility to graph the function.

19. $y = 45$ grams when $t = 0$; $y = 4$ grams when $t = 2$

20. $y = 75$ grams when $t = 0$; $y = 12$ grams when $t = 1$

In Exercises 21 and 22, use the Gompertz growth model described in Example 3 to find the growth function, and sketch its graph.

21. $L = 500$; $y = 100$ when $t = 0$; $y = 150$ when $t = 2$

22. $L = 5000$; $y = 500$ when $t = 0$; $y = 625$ when $t = 1$

23. ***Biology*** A population of eight beavers has been introduced into a new wetlands area. Biologists estimate that the maximum population the wetlands can sustain is 60 beavers. After 3 years, the population is 15 beavers. If the population follows a Gompertz growth model, how many beavers will be in the wetlands after 10 years?

24. ***Biology*** A population of 30 rabbits has been introduced into a new region. It is estimated that the maximum population the region can sustain is 400 rabbits. After 1 year, the population is estimated to be 90 rabbits. If the population follows a Gompertz growth model, how many rabbits will be present after 3 years?

Biology In Exercises 25 and 26, use the hybrid selection model in Example 4 to find the percent of the population that has the indicated characteristic.

25. You are studying a population of mayflies to determine how quickly characteristic A will pass from one generation to the next. At the start of the study, half the population has characteristic A. After four generations, 75% of the population has characteristic A. Find the percent of the population that will have characteristic A after 10 generations. (Assume $a = 2$ and $b = 1$.)

26. A research team is studying a population of snails to determine how quickly characteristic B will pass from one generation to the next. At the start of the study, 40% of the snails have characteristic B. After five generations, 80% of the population has characteristic B. Find the percent of the population that will have characteristic B after eight generations. (Assume $a = 2$ and $b = 1$.)

27. ***Chemical Reaction*** In a chemical reaction, a compound changes into another compound at a rate proportional to the unchanged amount, according to the model

$$\frac{dy}{dt} = ky.$$

(a) Solve the differential equation.

(b) If the initial amount of the original compound is 20 grams, and the amount remaining after 1 hour is 16 grams, when will 75% of the compound have been changed?

28. ***Chemical Mixture*** A 100-gallon tank is full of a solution containing 25 pounds of a concentrate. Starting at time $t = 0$, distilled water is admitted to the tank at the rate of 5 gallons per minute, and the well-stirred solution is withdrawn at the same rate.

(a) Find the amount Q of the concentrate in the solution as a function of t. (*Hint:* $Q' + Q/20 = 0$)

(b) Find the time when the amount of concentrate in the tank reaches 15 pounds.

29. ***Chemical Mixture*** A 200-gallon tank is half full of distilled water. At time $t = 0$, a solution containing 0.5 pound of concentrate per gallon enters the tank at the rate of 5 gallons per minute, and the well-stirred mixture is withdrawn at the same rate. Find the amount Q of concentrate in the tank after 30 minutes. (*Hint:* $Q' + Q/20 = \frac{5}{2}$)

30. ***Safety*** Assume that the rate of change in the number of miles s of road cleared per hour by a snowplow is inversely proportional to the depth h of snow. That is,

$$\frac{ds}{dh} = \frac{k}{h}.$$

Find s as a function of h if $s = 25$ miles when $h = 2$ inches and $s = 12$ miles when $h = 6$ inches $(2 \le h \le 15)$.

31. ***Chemistry*** A wet towel hung from a clothesline to dry loses moisture through evaporation at a rate proportional to its moisture content. If after 1 hour the towel has lost 40% of its original moisture content, after how long will it have lost 80%?

32. ***Biology*** Let x and y be the sizes of two internal organs of a particular mammal at time t. Empirical data indicates that the relative growth rates of these two organs are equal, and can be modeled by

$$\frac{1}{x}\frac{dx}{dt} = \frac{1}{y}\frac{dy}{dt}.$$

Use this differential equation to write y as a function of x.

33. ***Population Growth*** When predicting population growth, demographers must consider birth and death rates as well as the net change caused by the difference between the rates of immigration and emigration. Let P be the population at time t and let N be the net increase per unit time due to the difference between immigration and emigration. So, the rate of growth of the population is given by

$$\frac{dP}{dt} = kP + N, \quad N \text{ is constant.}$$

Solve this differential equation to find P as a function of time.

34. ***Meteorology*** The barometric pressure y (in inches of mercury) at an altitude of x miles above sea level decreases at a rate proportional to the current pressure according to the model

$$\frac{dy}{dx} = -0.2y$$

where $y = 29.92$ inches when $x = 0$. Find the barometric pressure (a) at the top of Mt. St. Helens (8364 feet) and (b) at the top of Mt. McKinley (20,320 feet).

35. ***Investment*** A large corporation starts at time $t = 0$ to invest part of its receipts at a rate of P dollars per year in a fund for future corporate expansion. Assume that the fund earns r percent interest per year compounded continuously. So, the rate of growth of the amount A in the fund is given by

$$\frac{dA}{dt} = rA + P$$

where $A = 0$ when $t = 0$. Solve this differential equation for A as a function of t.

Investment In Exercises 36–38, use the result of Exercise 35.

36. Find A for each situation.

(a) $P = \$100{,}000$, $r = 12\%$, and $t = 5$ years

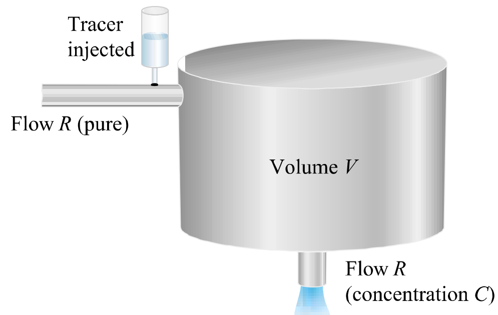
(b) $P = \$250{,}000$, $r = 15\%$, and $t = 10$ years

37. Find P if the corporation needs \$120,000,000 in 8 years and the fund earns $16\frac{1}{4}\%$ interest compounded continuously.

38. Find t if the corporation needs \$800,000 and it can invest \$75,000 per year in a fund earning 13% interest compounded continuously.

Medical Science In Exercises 39–41, a medical researcher wants to determine the concentration C (in moles per liter) of a tracer drug injected into a moving fluid. Solve this problem by considering a single-compartment dilution model (see figure). Assume that the fluid is continuously mixed and that the volume of fluid in the compartment is constant.

Figure for 39–41



39. If the tracer is injected instantaneously at time $t = 0$, then the concentration of the fluid in the compartment begins diluting according to the differential equation

$$\frac{dC}{dt} = \left(-\frac{R}{V}\right)C, \quad C = C_0 \text{ when } t = 0.$$

(a) Solve this differential equation to find the concentration as a function of time.

(b) Find the limit of C as $t \to \infty$.

40. Use the solution of the differential equation in Exercise 39 to find the concentration as a function of time, and use a graphing utility to graph the function.

(a) $V = 2$ liters, $R = 0.5$ L/min, and $C_0 = 0.6$ mol/L

(b) $V = 2$ liters, $R = 1.5$ L/min, and $C_0 = 0.6$ mol/L

41. In Exercises 39 and 40, it was assumed that there was a single initial injection of the tracer drug into the compartment. Now consider the case in which the tracer is continuously injected (beginning at $t = 0$) at the rate of Q mol/min. Considering Q to be negligible compared with R, use the differential equation

$$\frac{dC}{dt} = \frac{Q}{V} - \left(\frac{R}{V}\right)C, \quad C = 0 \text{ when } t = 0.$$

(a) Solve this differential equation to find the concentration as a function of time.

(b) Find the limit of C as $t \to \infty$.