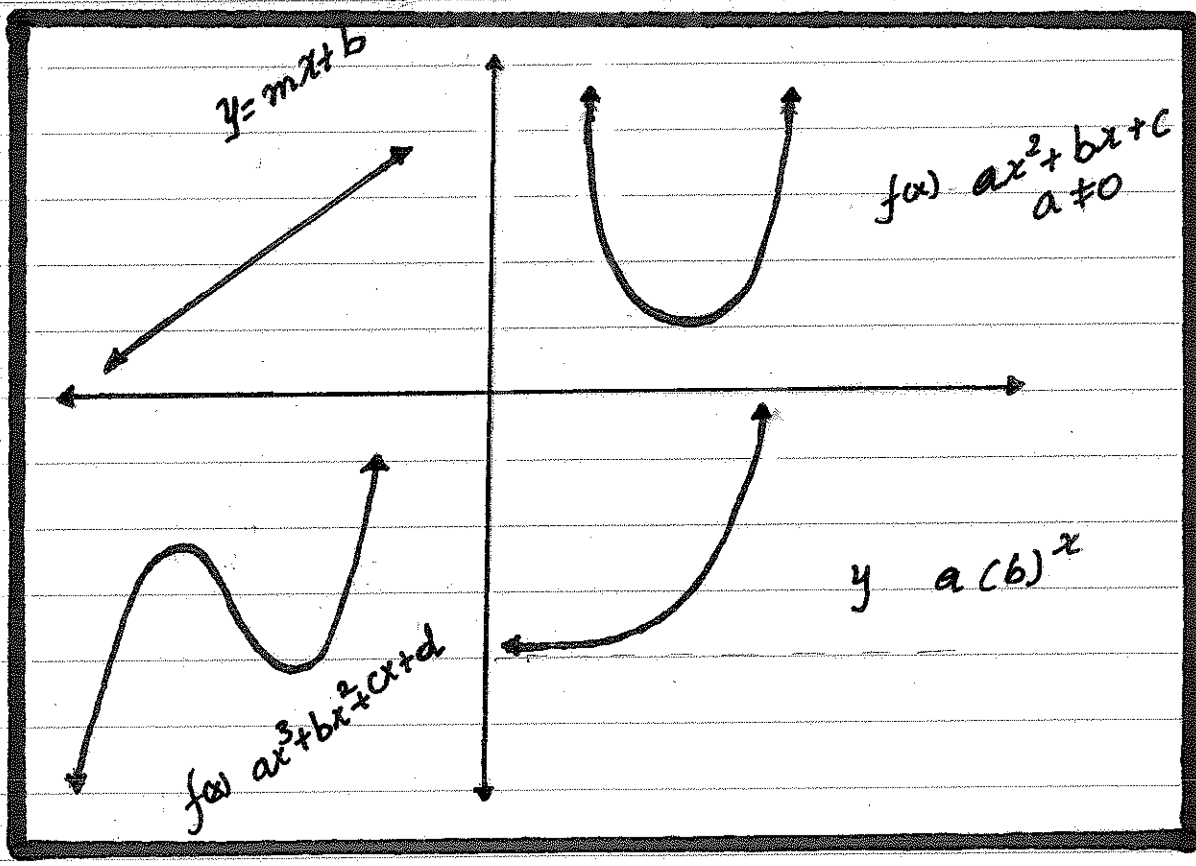
ALGEBRA I



FIRST EDITION

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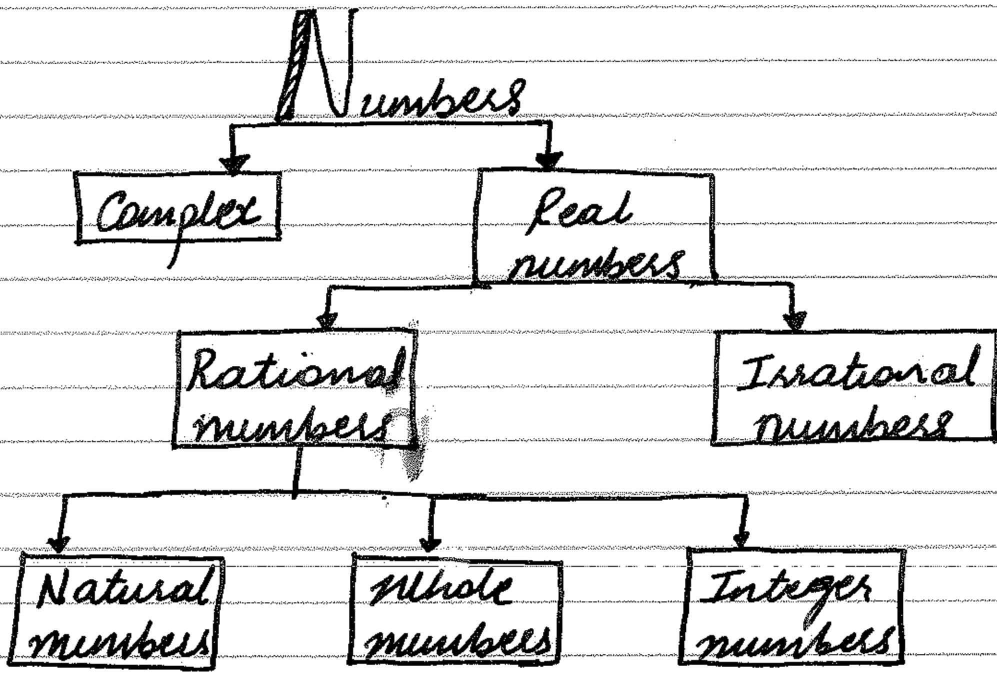
why math

At some point in every child's life, there is a time where math made sense and hence fun. It might be adding numbers or drawing shapes. It might be solving problems or modeling situation, in any case it was at least interesting. So, where did it become less interesting or even challenging?

Most children share the same experience interms of challenging math. Once variables are introduced to the math world, things start to get challenging and hence difficult to understand.

Naturally, we tend to understand concrete objects, where we can actually touch and feel. a good example would be counting objects. The next thing will be visualizing objects. A good example would be coloring objects, i.e catagorizing object with similar color/properties. The third piece of this puzzle would be the abstract part. Normally, variables are used to represent abstract part of math.

Concrete Objects → pictorial objects → Abstract objects



Even numbers:
Integers that can be divided by 2 with out a remainder

Odd numbers:
Integers that are not divisible by 2.

Prime numbers:
Are natural numbers greater than 1 that has only two divisor i.e 1 and itself.
2 is the only even prime number.

Composite numbers:
Are natural numbers that have at least one divisor other than 1 and themselves.

Operations.

Addition
Subtraction
multiplication
Division

Basic mathematical operations

Parentesis
exponents

Other / common mathematical operations

Symbols
(+) addition
(−) subtraction
(∗) multiplication
(÷) or (/) division
(( )) parentesis

Addition
multiplication
Parentesis
PEMDAS
Subtraction
Division
Exponents

Order of Operation

Algebraic Expression

Numbers
Coefficients
Variables

Example

$2x^2 + 1$

Algebraic Equation

Two algebraic expressions connected by the equal sign.

Example

$2x^2 + 1 = x + 1$

Evaluate is for an algebraic expression.
SOlve is for an algebraic equation.

Examples

① Evaluate the expression

$2x^2+1$ for $x = 2$

$x = -5$

Solution

$2x^2+1$

$2(2)^2+1$

$2(4)+1$

$8+1$

9

$2x^2+1$

$2(-5)^2+1$

$2(25)+1$

$50+1$

51

② Evaluate the expression

$-3x+2(x+2y)-5$ for a) $x=1$ & $y=-2$

b) $x=3$ & $y=5$

a) $-3x+2(x+2y)-5$

$-3(1)+2(1+2(-2))-5$

$-3+2(1-4)-5$

$-3+2(-3)-5$

$-3+(-6)-5$

$-3-6-5$

-14

b) $-3x+2(x+2y)-5$

$-3(3)+2(3+2(5))-5$

$-9+2(3+10)-5$

$-9+2(13)-5$

$-9+26-5$

$-9+21$

12

③ Solve the equation

$2x+5=-21$

$-5 \quad -5$

$\frac{2x}{2} = \frac{-26}{2}$

$x=-13$

$-2x-3(x+2) = 19$

$-2x-3x-6 = 19$

$-5x-6 = 19$

$+6 \quad +6$

$\frac{-5x}{-5} = \frac{25}{-5}$

$x=-5$

Fractions

Fractions are any number that can be written as $\frac{a}{b}$; $a, b \in \mathbb{Z}$ & $b \neq 0$
↳ Integers

Examples: $\frac{2}{3}, \frac{1}{2}, \frac{-3}{5}, \frac{5}{1}, \ldots$

Arithmetic & Fractions

Multiplying two fractions

notation $\frac{a}{b}$ $a \rightarrow$ numerator, $b \rightarrow$ denominator

When we multiply two fraction, we multiply the numerator of the 1st fraction with the numerator of the 2nd fraction and we do the same thing with their denominators:

Eg1. $\frac{3}{4} * \frac{5}{6} = \frac{3*5}{4*6} = \frac{15}{24}$ Now we can simplify

$$\Rightarrow \frac{15}{24} = \frac{5}{8}$$

Eg2. $\frac{-1}{2} * \frac{4}{3}$

$$= \frac{-1*4}{2*3} = \frac{-4}{6}$$ simplify $\frac{-4}{6} = \frac{-2}{3}$

Eg3. $\frac{2}{5} * \frac{4}{6} * \frac{-1}{2}$

$$= \frac{2*4*-1}{5*6*2} = \frac{-8}{60}$$ Simplify $\frac{-8}{60} = \frac{-2}{15}$

Dividing two fractions:

Dividing two fraction is one more step compared with multiplying two fractions.

Key word: Reciprocal

the reciprocal of any fraction is swapping the numerater by the dinominater.

Eg: $\frac{2}{5} \rightarrow$ Reciprocal will be $\frac{5}{2}$

☆ Dividing by a fraction is the same as multiplying by the reciprocal of the divisor:

Eg 1. $\frac{5}{3} \div \frac{10}{6}$ swap

$$= \frac{5}{3} * \frac{6}{10} = \frac{5*6}{3*10} = \frac{30}{30} = \frac{1}{1} = 1$$

Eg 2. $$\frac{\frac{4}{5}}{\frac{2}{15}} \Rightarrow \frac{4}{5} \div \frac{2}{15}$$

$$= \frac{4}{5} * \frac{15}{2} = \frac{4*15}{5*2} = \frac{60}{10} = \frac{6}{1} = 6$$

Eg 3. $1/3 \div \frac{2}{5}$

$$\Rightarrow 1/3 * \frac{5}{2} = \frac{1*5}{3*2}$$

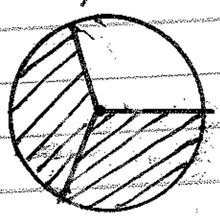
$$= \frac{5}{6}$$

Subtracting / Adding fractions

In order to understand adding / subtracting fractions, 1st let's try to compare two fractions.

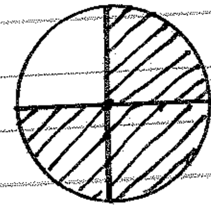
$\frac{2}{3}$ and $3/4$

These two fractions don't have a common dinominator, the 1st fraction ($2/3$) tells us 2 out of 3 and the 2nd fraction ($3/4$) tells us 3 out of 4.



$2/3$

Unless both fractions are split among equal parts, it's very difficult to compare.



$3/4$

Eg.

Assume you have 50 U.S dollar and your friend has 150 chinese yen. The question is, who has more money in England, if you both were there at the same time.

we can not answer / tell for sure whom has more unless we convert your money into pounds.

1 pound ≅ 150 yen

1 pound ≅ 2 USD

⇒ you have about 25 pounds

your friend has about 1 pound

Now you can easily tell that you have more money.

* The same thing works when we compare fractions. We need to have the SAME DINOMINATORS.

$\frac{2}{3}$ the dinominator is 3

$\frac{3}{4}$ the dinominator is 4

there a few different techinques that can be used to create a common dinominator.

1st: Finding the LCM (least common multiple)

This is how it done:

multipliers:	(x)	1	2	3	4	5	----
1st denominator:	3:	3	6	9	$\boxed{12}$	15	
2nd denominator:	4:	4	8	$\boxed{12}$	16	20	

the LCM of 3 & 4 is 12.

LCM (3, 4) = 12.

1st fraction $\frac{2}{3}$ need to be multiplied by $\frac{4}{4}$ and

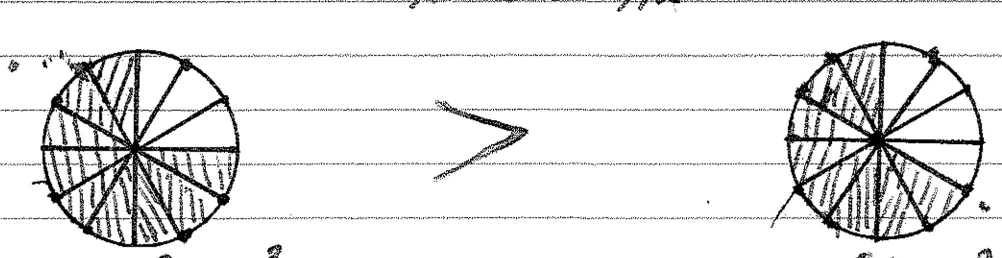
2nd fraction $\frac{3}{4}$ need to be multiplied by $\frac{3}{3}$.

Remember that $4/4$ & $3/3$ equals 1, so we are not affecting the integrity of both fractions, we are converting them into equivalent fractions:

$\frac{2}{3} \times \frac{4}{4} = \frac{8}{12}$ $\qquad$ $\frac{3}{4} \times \frac{3}{3} = \frac{9}{12}$

Now we can compare

$8/12$ and $9/12$



$9/12 = \frac{3}{4}$ $\qquad$ $8/12 = \frac{2}{3}$

Eg1.

$\frac{2}{3} + \frac{3}{4}$, notice that they don't have a common dinominator

$\Rightarrow$ multiply both fractions by the opposite dinominators i.e

$\frac{2}{3} * \frac{4}{4} + \frac{3}{4} * \frac{3}{3}$ $\quad$ $\frac{2}{3}$ by $4/4$ and $3/4$ by $3/4$

$\frac{8}{12} + \frac{9}{12}$ $\quad$ now they share a common dinominator so, take one of the dinominators and Combine the numerators i.e

$$\frac{8+9}{12}$$

$= \frac{17}{12}$ $\quad$ since 17 is a prime, we can NOt further simplify the result.

Eg2.

$\frac{8}{6} - \frac{3}{5}$

1st find their common dinominator i.e multiply $\frac{8}{6}$ by $\frac{5}{5}$ & $\frac{3}{5}$ by $\frac{6}{6}$.

$\frac{8}{6} * \frac{5}{5} - \frac{3}{5} * \frac{6}{6}$

$= \frac{40}{30} - \frac{18}{30}$

$= \frac{40-18}{30}$

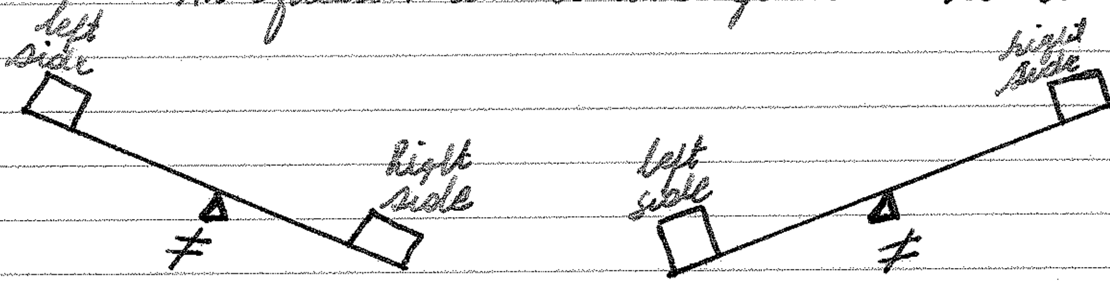
$= \frac{22}{30}$ $\quad$ now simplify the fraction

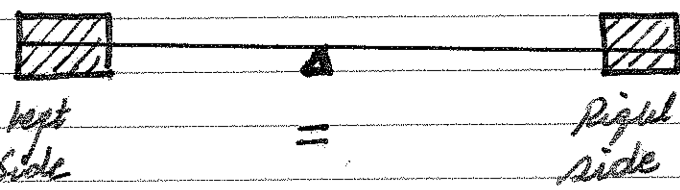
$= \frac{11}{15}$

Solving Equations

Any equation has two sides separated by an equal sign. left side of an equation and right side of an equation.

An equation can be analized like see saw swing





When an equation is given, both the left & right sides of the equation are balanced. It is very important to preserve their properties.

what ever change we make to either side of the equation, must be done to the other side of the equation.

An equation normal has the following:

- Variables
- operations
- Numbers
- Equal sign

when we solve an equation (single variable) our ultimate goal is to find a value / values of a variable that makes/make the equation true.

One way to do this is, to leave the variable on one side of the equation alone and take everything to the other side by following the proper mathematical steps & rules.

When we evaluate/simplify an expression, we follow the rule of "PEMDAS". Now, when we solve an equation, we can follow the exact opposite route. "SADMEP".

Examples:

Eg1.

$$\underbrace{x+4}_{\text{left side}} = \underbrace{10}_{\text{right side}}$$

we want to leave x by itself on the left side of the equation.
⇒ subtracting 4 from both sides

$$x+4 = 10$$
$$-4 \quad -4$$

$x = 6$ ⟶ This is also called one step equation solving.

Eg2.

$$2x+4 = 10$$

⟶ we want to leave x by itself on the left side. notice that 4 is connected with x by addition & 2 is connected with x by multiplica

$$2x+4 = 10$$
$$-4 \quad -4$$
$$\frac{2x}{2} = \frac{6}{2}$$

$x = 3$ ⟶ Two steps equation solving

Eg3. $\frac{x}{2} + 4 = 10$

$$\frac{x}{2} + 4 = 10$$
$$-4 \quad -4$$
$$2 \times \frac{x}{2} = 6 \times 2$$

$$x = 12$$

Multi-step & Equations with variables on both sides of the equal sign.

Before we try to solve a value of a variable that makes a given equation true, we need to make sure that all the variables present in the equation are combined together with the proper mathematical steps:

Combining like terms:

$2x + 3x$	$-3x + x$	$1+4$	$2x-1+x$
$5x$	$2x$	5	$3x-1$

Eg1.

$2x + 1 = 5x - 5$ — 1st we need to combine the like terms i.e $2x$ & $5x$, 1 & -5

$2x + 1 = 5x - 5$
$\quad -1 \qquad -1$

$2x = 5x - 6$
$-5x \quad -5x$ — now we can divide both sides by "-3" to get the value of 'x'.

$-3x = -6$

$\dfrac{-3x}{-3} = \dfrac{-6}{-3}$

$\underline{\underline{x = 2}}$

Eg2: $2(-3x+1) - 2 = 3x - 4(2+x)$

1st distribute the 2 on the left side and -4 on the right side over the parentesis.

$2(-3x+1) - 2 = 3x - 4(2+x)$

$-6x + 2 - 2 = 3x - 8 - 4x$ • Combine like terms

$-6x = -1x - 8$ • add 1x to both sides

$+1x \quad +1x$

$\dfrac{-5x}{-5} = \dfrac{-8}{-5}$ $\quad x = 8/5 = 1\frac{3}{5}$

When we have relatively complex equations, we need to simplify each of the sides before we combine like terms.

Simplifying always follow PEMDAS

Solving equation is the reverse process of simplification.

Eg3. $\frac{1}{2}x + 2(x-1) = \frac{2}{5}x - 3$

distribute 2 over $(x-1)$

$\frac{1}{2}x + 2x - 2 = \frac{2}{5}x - 3$

$\frac{1}{2}x + \frac{4}{2}x - 2 = \frac{2}{5}x - 3$

• Now we can combine $\frac{1}{2}x$ & $2x$, remember $2x = \frac{4}{2}x$

$\frac{1+4}{2}x - 2 = \frac{2}{5}x - 3$

$\frac{5}{2}x - 2 = \frac{2}{5}x - 3$ • add 2 to both sides

$\quad +2 \qquad +2$

$\frac{5}{2}x = \frac{2}{5}x - 1$ • subtract $\frac{2}{5}x$ from both sides

$-\frac{2}{5}x \qquad -\frac{2}{5}x$

$\frac{5}{2}x - \frac{2}{5}x = -1$ • to subtract $\frac{2}{5}$ from $\frac{5}{2}$ we need to have the same denominator.

$\left(\frac{5}{5}\right)\frac{5}{2}x - \left(\frac{2}{2}\right)\frac{2}{5}x = -1$

$\frac{25}{10}x - \frac{4}{10}x = -1$

$\frac{25-4}{10}x = -1$

• multiply both sides by $\frac{10}{21}$ to get x by itself!

$\frac{21}{10}x = -1$

$\left(\frac{10}{21}\right)\frac{21}{10}x = -1\left(\frac{10}{21}\right)$

$x = -\frac{10}{21}$ //

Inequalities:

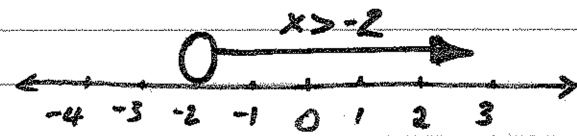
Equations, normally have a specific value/values for their variable that make them true.
Inequalities don't have a specific value, rather they have a <u>set</u> of possible values, that make the whole inequality true.

<u>Inequality signs</u>

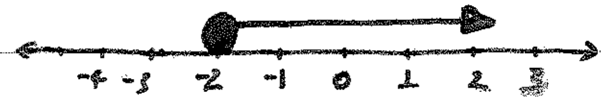
$>$ "greater than"
$\geq$ "greater than or equal to"
$<$ "less than"
$\leq$ "less than or equal to"

After solving Inequalities, it is customary to show the solution on a number line.

Eg.
let $x > -2$ is a solution.



$x \geq -2$



solid circles: $\leq$ $\geq$ open circles: $<$, $>$

* Open circle means, the solution does NOT include the start point but everything above/below it.

* Closed circle means, the solution INCLUDES the starting point and the rest.

When we multiply or divide inequalities by a negative number, the inequality sign must also be multiplied / divided by negative, which reverses / switch the sign.

i.e $- > \; = \; <$ $\qquad - < \; = \; >$

$- \geq \; = \; \leq$ $\qquad - \leq \; = \; \geq$

Solving inequalities:

Eg 1. $x + 4 > 2$

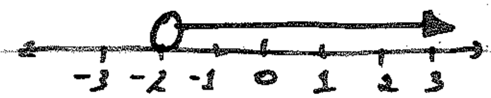
$x + 4 > 2$ · subtract 2 from both sides

$\quad -2 \quad -2$

$x + 2 > 0$ · add -2 to both sides

$\quad -2 \quad -2$

$x > -2$



Eg 2. $3 - 2x \leq -7$

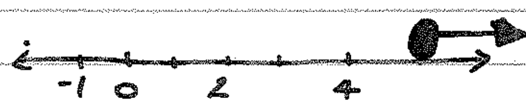
$3 - 2x \leq -7$: subtract 3 from both sides

$-3 \qquad\quad -3$

$-2x \leq -10$ · divide both side by -2

$\frac{-2x}{-2} \leq \frac{-10}{-2}$ * remember dividing by -2 switch the inequality sign.

$x \geq 5$



Eg 3. $2x + 3 \leq x$

$2x + 3 \leq x$ · subtract 2x from both sides

$-2x \qquad -2x$

$\frac{3}{-1} \leq \frac{-x}{-1}$ · divide both sides by -1, switch the sign.

$-3 \geq x$

$x \leq -3$

Multi-step Inequality

Eg1.

$2(3x-5)+4 \le -3x+3(x-1)$

- 1st distribute the 2 & 3 over the parenthesis

$2(3x-5)+4 \le -3x+3(x-1)$

$6x-10+4 \le -3x+3x-3$

- combine like terms on each sides

$6x-6 \le -3$

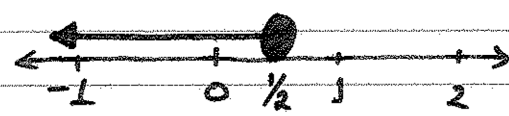
$6x-6 \le -3$ · add 6 to both sides

$+6 \quad +6$

$6x \le 3$ · divide by 6 both sides

$\frac{6x}{6} \le \frac{3}{6}$

$x \le 1/2$



Eg2.

$-2(x-1)+3x-1 \ge -2(4x+1)+9x$

$-2(x-1)+3x-1 \ge -2(4x+1)+9x$

$-2x+2+3x-1 \ge -8x-2+9x$

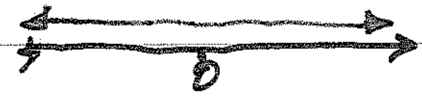
$-2x+3x+2-1 \ge -8x+9x-2$

$x+1 \ge x-2$

$x+1 \ge x-2$ subtract x from both sides

$-x \qquad -x$

$1 \ge -2$ → This statement is always true therefore the solution set is everything



Compound Inequalities

Compound inequalities are inequalities where more than one inequality exists.

Dependending on the type of inequalities, compound inequalities may be combined by "AND" or "OR"

Eg: $2x-1 \leq 3x+5 \leq 2x+11$

2: $5 < x+2 < 12$

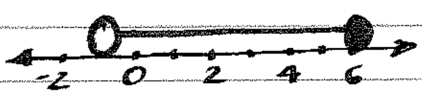
3: $-3 > 2x-1 \geq 4$

Solving compound inequalities

Eg 1: $-2 < x-1 \leq 5$

$+1 \quad +1 \quad +1$ add 1 to everything

$-1 < x \leq 6$



S.S. = $\{-1 < x \leq 6\}$

Alternative method

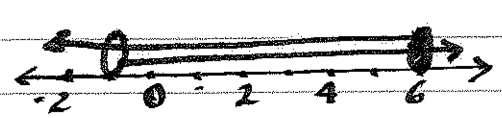
Split the compound inequality

$-2 < x-1 \leq 5$

$-2 < x-1$ AND $x-1 \leq 5$

$+1 \quad +1$ AND $+1 \quad +1$

$-1 < x$ AND $x \leq 6$



The solution is the intersection of the two inequalities

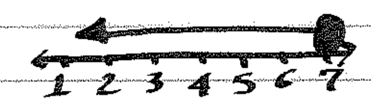
S.S = $\{-1 < x$ And $x \leq 6\} = \{-1 < x \leq 6\}$

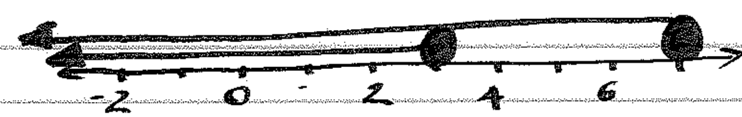
AND is equivalent to Intersection.
OR is equivalent to Union.

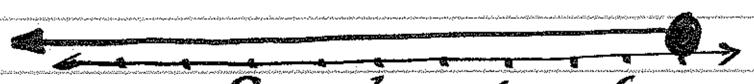
Eg 2: $3x-5 \le 4$ OR $4-2x \ge 10$

$$
\begin{array}{lcl}
3x-5 \le 4 & \quad & 4-2x \ge 10 \\
\ \ +5 \quad +5 & & -4 \qquad -4 \\
\dfrac{3x}{3} \le \dfrac{9}{3} & & \dfrac{-2x}{-2} \ge \dfrac{-14}{-2} \\
x \le 3 & & x \le 7
\end{array}
$$

Remember to switch the inequality sign when dividing by negative #'s.







Since the compound inequalities are connected by "OR" the solution is going to be the "Union" of the two individual solutions.

$S.S. = \{ x \le 7 \}$

OR

If you want at answer in an interval notation

$S.S. = \{ [-\infty, 7] \}$

Relations and functions

A relation is defined as a set of ordered pairs.
A set is a collection of things.

Eg: a set is denoted by the symbol { }
R = { (Brother, sister), (Boy, girl) }
R = { (1,2), (4,5) }
R = { (homes, money), (food, money), (love, money) }

notice that each relation contains an ordered pair/pairs.

Remark:
The first entry of an ordered pair is commonly known as an Inpute.
The second entry is known as an Output.

A function is a relation where each input is paired with exactely one distinct out put.

Eg: 1
$f = \{ (1,2), (3,4), (5,6) \}$

notice that the inputs of this relation $\{1,3,5\}$

notice the out puts of the relation $\{2,4,6\}$

each input is paired with exactely one unique output
1 paired with 2
3 paired with 4
5 paired with 6.
Therefore; f is a function.

Eg: 2
$g = \{ (1,2), (3,4), (1,5) \}$

input = $\{1,3\}$
output = $\{2,4,5\}$

1 is paired with 2
1 is also paired with 5
3 is paired with 4

Since 1 is paired with two different outputs, g is NOT a function.

More Examples:

3. $f = \{(1,4), (-2,3), (5,4)$

1 is paired with 4

-2 is paired with 3

5 is paired with 4

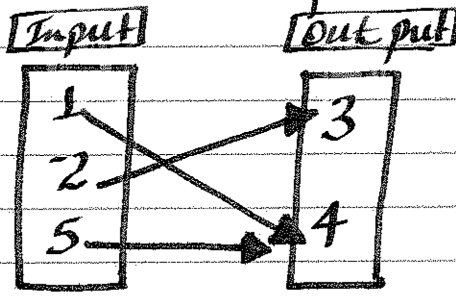
each input is paired with one unique output. Notice that 1 and 5 are paired with 4 and that is okay,

$\therefore f$ is a function!

This example shows that an output can have multiple input, but an input can NOT have more than one output & be a function.

Venn Diagram for Input / Output

The above example can be represented by

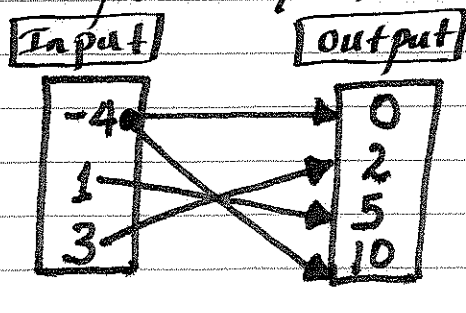


Function

4 $g = \{(-4,0), (1,5), (-4,10), (3,2)\}$

Input = $\{-4, 1, 3\}$

Output = $\{0, 2, 5, 10\}$

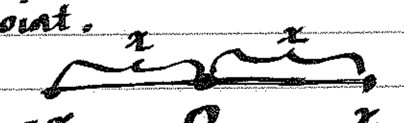


Since -4 is paired with two different output, (0 & 10), g is NOT a function

Absolute Values.

Absolute values are represented by two long parallel lines "| |".

Absolute values produce positive output.
It can be seen as the distance from a fixed point.

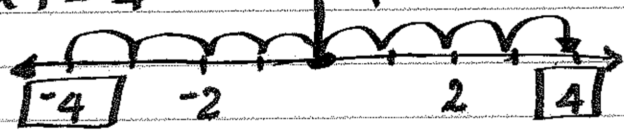


$|x|$ is always positive

starting position

Eg 1:

$|x| = 4$



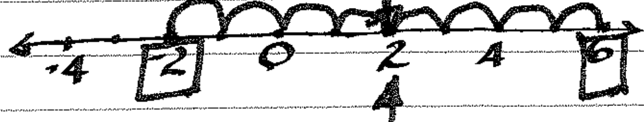
Absolute value can also be defined as distance on a number line.

moving forward } distance is
moving backward } always positive.

Eg 2: $|x - 2| = 4$

the starting position is found by setting whatever is inside the absolute value sign to 0.

$$x - 2 = 0$$
$$+2 \quad +2$$
$$x = 2$$ starting position



S.S. = {-2, 6}

Remark:

Anything that comes out of an absolute value is always positive

Eg:

$|4| = 4$ $\quad |-x| = x$

$|-4| = 4$ $\quad |-2| = 2$

Solving absolute equations:

whenever you solve absolute value equations, you need to consider "TWO" cases:

$\boxed{1^{st}\text{ case:}}$

Treat the absolute value sign like a parenthesis and solve

$\boxed{2^{nd}\text{ case:}}$

- Change the signs of everything inside the absolute value sign and solve.

Eg 1. $|x+2| = 3$

Case 1

$|x+2| = 3$

$(x+2) = 3$

$x+2 = 3$

$\quad -2 \quad -2$

$x = 1$

Case 2

$|x+2| = 3$

$(-x-2) = 3$

$-x - 2 = 3$

$\quad +2 \quad +2$

$\frac{-x}{-1} = \frac{5}{-1}$

$x = -5$

$S \cdot S = \{-5, 1\}$

Check:

$|x+2| = 3$

$|-3+2| = 3$

$|-3| = 3$

$3 = 3$ ✓

$|x+2| = 3$

$|1+2| = 3$

$|3| = 3$

$3 = 3$ ✓

MORE Examples:

2. $2|x-3|-4=6$

Case 1:

$2(x-3)-4=6$

$2x-6-4=6$

$2x-10=6$

$\quad +10 \quad +10$

$\frac{2x}{2}=\frac{16}{2}$

$x=8$

Case 2:

$2(-x+3)-4=6$

$-2x+6-4=6$

$-2x+2=6$

$\quad -2 \quad -2$

$\frac{-2x}{-2}=\frac{4}{-2}$

$x=-2$

Check

$2|x-3|-4=6$

$2|8-3|-4=6$

$2|5|-4=6$

$10-4=6$

$6=6$ ✓

$2|x-3|-4=6$

$2|-2-3|-4=6$

$2|-5|-4=6$

$2(5)-4=6$

$10-4=6$

$6=6$ ✓

S.S. $=\{x=-2,8\}$

3. $\frac{1}{2}|2x+1|+2=5$

Case 1:

$\frac{1}{2}(2x+1)+2=5$

$x+\frac{1}{2}+2=5$

$x+\frac{5}{2}=5$

$\quad -\frac{5}{2} \quad -\frac{5}{2}$

$x=5-\frac{5}{2}$

$=\frac{10}{2}-\frac{5}{2}$

$=\frac{10-5}{2}$

$x=\frac{5}{2}$

Case 2:

$\frac{1}{2}(-2x-1)+2=5$

$-x-\frac{1}{2}+2=5$

$-x+\frac{3}{2}=5$

$\quad -\frac{3}{2} \quad -\frac{3}{2}$

$-x=5-\frac{3}{2}$

$=\frac{10}{2}-\frac{3}{2}$

$=\frac{10-3}{2}$

$-x=\frac{7}{2}$

$*-1 \quad *-1$

$x=-\frac{7}{2}$

S.S. $=\{x=-\frac{7}{2},\frac{5}{2}\}$

Eg 4:

$2|2x-10| = -2$

By just looking at the Right side of the equation, we can tell that there is <u>NO</u> sol^n. This is because, nomatter what x we can find, when we put it in the left side, it will always be positive and that doesn't equal -2.

let's Solve

Case 1:

$2(2x-10) = -2$

$4x - 20 = -2$

$+20 \quad +20$

$\frac{4x}{4} = \frac{18}{4}$

$x = 4.5$

Case 2:

$2(-2x+10) = -2$

$-4x + 20 = -2$

$-20 \quad -20$

$\frac{-4x}{-4} = \frac{-22}{-4}$

$x = 5.5$

Check:

$2|2x-10| = -2$

$x = 4.5$

$2|2(4.5)-10| = -2$

$2|9-10| = -2$

$2|-1| = -2$

$2(1) = -2$

$2 = -2$ False

$x = 5.5$

$2|2(5.5)-10| = -2$

$2|11-10| = -2$

$2|1| = -2$

$2(1) = -2$

$2 = -2$ False

As a result the solution set is an Empty Set.

$S.S. = \{\phi\}$

Symbol for Empty set

Absolute Value Inequalities

An absolute value inequality is similar to an absolute value equation in many ways.

When we solve an inequality, we need to make sure that our solution set is a set of possible values that make the inequality true.

Important:
whenever we multiply or divide an inequality by a negative number, the inequality sign **switches**.

Eg1:

$-x > 5$

$\times -1 \qquad \times -1$

$x < -5$

Eg2. $-x \leq -2$

$\div -1 \qquad \div -1$

$x \geq 2$

Notice that in both cases the inequality sign switched after multiplying or dividing by a negative number.

Solving absolute value Inequalities

Eg1

$|x-2| < 3$

Case 1

$(x-2) < 3$

$x-2 < 3$

$+2 \quad +2$

$x < 5$

AND

Case 2

$(-x+2) < 3$

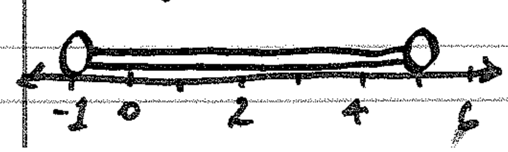
$-x+2 < 3$

$-2 \quad -2$

$\frac{-x}{-1} < \frac{1}{-1}$

$x > -1$

$S.S = \{x : x < 5 \text{ And } x > -1\}$



More Examples:

Eg 2.

$$2|x-6| \le 2$$

Case 1:

$2(x-6) \le 2$

$2x - 12 \le 2$

$\quad +12 \quad +12$

$\frac{2x}{2} \le \frac{14}{2}$

$x \le 7$

AND

Case 2:

$2(-x+6) \le 2$

$-2x + 12 \le 2$

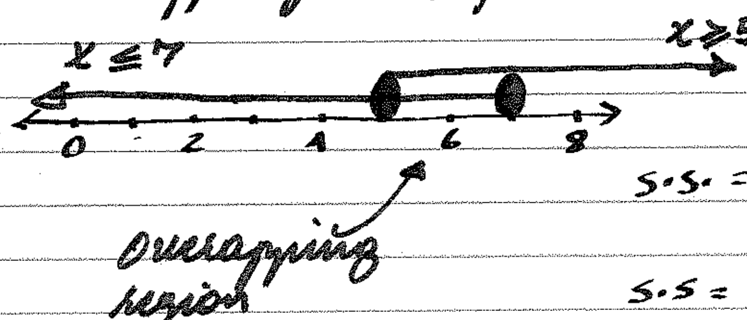
$\quad -12 \quad -12$

$\frac{-2x}{-2} \le \frac{-10}{-2}$

$x \ge 5$

↳ notice that the inequality sign switched when dividing by -2.

* And implies that the sol^n is an intersection of the two cases.
* On a number line, it is the overlapping set of points.



$s.s. = \{x : x \ge 5 \text{ And } x \le 7\}$

OR

$s.s = \{x : 5 \le x \le 7\}$

OR

$s.s. = \{x : [5, 7]\}$

Remark

It is always important to remember to check you sol^n set.

* Sometime, even if we can solve the inequalities, the sol^n doesn't work

Check:

let $x = 6$

6 is b/n 5 & 7

$2|x-6| \le 2$

$2|6-6| \le 2$

$2|0| \le 2$

$0 \le 2$ ✓

Eg 3:

Before we conclude that the solution set for a given absolute value inequality, it is very important to check the answers.

For example:

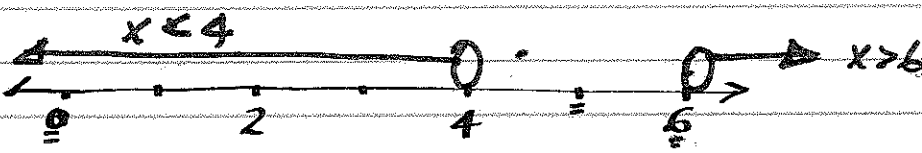
$4|2x-10| < -8$

Does Not Have a sol^{n}

$4|2x-10| < -8$

Case 1	Case 2
$4(2x-10) < -8$	$4(-2x+10) < -8$
$8x-40 < -8$	$-8x+40 < -8$
$+40 \quad +40$	$-40 \quad -40$
$\frac{8x}{8} < \frac{32}{8}$	$\frac{-8x}{-8} < \frac{-48}{-8}$
$x < 4$	$x > 6$

$S.S = \{\phi\}$



From the graph, we can see that the two cases don't have any common intersection.

As a result there is no sol^{n} for the above absolute value inequality.

Check for $x=0 \quad x=6 \quad x=5$

$4|2x-10| < -8$

$4|2(0)-10| < -8$

$4|0-10| < -8$

$4|-10| < -8$

$4(10) < -8$

$40 < -8$ False

$4|2(5)-10| < -8$

$4|0| < -8$

$0 < -8$ False

$4|2(6)-10| < -8$

$4|12-10| < -8$

$4|2| < -8$

$8 < -8$ False

Ratio

A ratio is a comparison between two numbes by division.

Notation: " : "
" / "

For example
- 4:5 read as four to five.
- $\frac{4}{5}$ " " " " " .

A Unit rate:

A unit rate is when we compare a number with 1.

i.e $\frac{12}{3}$ is NOT a unit rate. This is because 12 is compared with 3 not 1.

If we want to change $\frac{12}{3}$ into a unit rate, we need to divid 12 by 3.

$\frac{12}{3} = \frac{4}{1}$ Now $\frac{4}{1}$ is a unit rate

Change $\frac{5}{2}$ into a unit rate:

$\frac{5}{2} = 5 \div 2 = \frac{2.5}{1}$ unit rate $2.5/1$

Eg. Let 4 once of tomatos cost 5\$
what is the unit rate?

$\frac{4\ oz}{5\ \$} = 40 \div 50\ \frac{oz}{\$}$

$= \frac{0.8}{1}\ \frac{oz}{\$}$

$= 0.8\ oz/\$$

The unit rate implies that for ever dollar, we spent, we get 0.8 oz tomatos.

Working with proportions

A proportion is comparison between two ratios by the equal sign

Eg: $\frac{1}{2} = \frac{4}{8}$ read as: one half is proportional to 4, 8th.

In general a proportion is represented by: $\frac{a}{b} = \frac{c}{d}$ where $a, b, c, d \in \mathbb{Z}$ and $b \neq 0$, and $d \neq 0$

a and d are <u>extremes</u> of the proportion &
b and c are <u>means</u> of the proportion.

To solve proportions, we can use cross product and multiplication property of equality.

Eg:
$$\frac{x}{4} = \frac{2}{8}$$

cross product:
extremes product = means product

$$\frac{\boxed{x}}{4} = \frac{2}{\boxed{8}}$$

$$(x)*(8) = (4)*(2)$$

$$\frac{8x}{8} = \frac{8}{8}$$

$$\boxed{x = 1}$$

Eg: $\frac{x+2}{3} = \frac{6}{2}$

$$(2)(x+2) = (3)(6)$$

$$2x + 4 = 18$$
$$-4 \quad -4$$

$$\frac{2x}{2} = \frac{14}{2}$$

$$\boxed{x = 7}$$

Check

$$\frac{x+2}{3} = \frac{6}{2}$$

$$\frac{7+2}{3} = \frac{6}{2}$$

$$\frac{9}{3} = \frac{6}{2}$$

$$3 = 3 \checkmark$$

Eg
Solve the proportion

$$\frac{2x+1}{4} = \frac{x+5}{2}$$

extremes product = means product

$$(2)(2x+1) = (4)(x+5)$$

$$4x+2 = 4x+20$$
$$\quad -2 \qquad\quad -2$$

$$4x = 4x+18$$
$$-4x \quad -4x$$

$0 = 18$ False Therefore; the is no such 'x' that makes the proportion true:

No solution

Eg solve the proportion

$$\frac{6}{3x+6} = \frac{2}{x+2}$$

$$(6)(x+2) = (2)(3x+6)$$

$$6x+12 = 6x+12$$
$$\quad -12 \qquad\quad -12$$

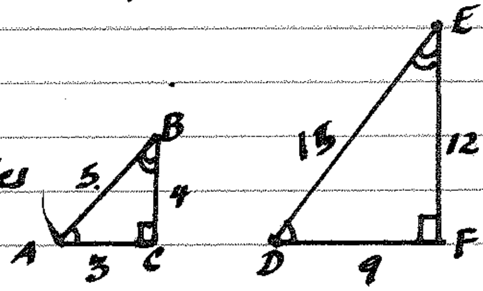
$$6x = 6x+0$$
$$-6x \quad -6x$$

$0 = 0$ True

This implies that the proportion is an Identity. Therefore; any 'x' value makes the proportion true $x = \mathbb{R}$.

Similar Triangles and Application of proportions

Similar triangles have the same angles and proportional dimensions / sides



Symbol for similarity
($\sim$)

$\triangle ABC \sim \triangle DEF$

because $\angle A \cong \angle D$ And $\frac{DF}{AC} = \frac{9}{3} = \boxed{3}$

($\cong$) congruent $\quad \angle B \cong \angle E$

$\angle C \cong \angle F$ $\quad \frac{EF}{BC} = \frac{12}{4} = \boxed{3}$

$\frac{DE}{AB} = \frac{15}{5} = \boxed{3}$

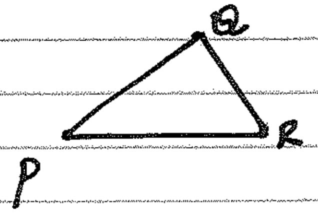
All the angles are congruent and the sides are proportional.

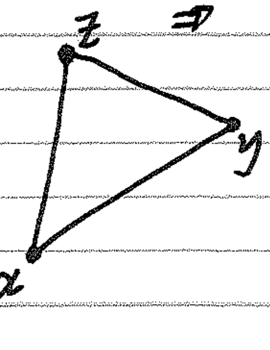
$\Rightarrow \triangle ABC \sim \triangle DEF$

It is very important to write the similarity statement in the right order. i.e $\angle A$ must match with $\angle D$
$\angle B$ " " " $\angle E$ &
$\angle F$ " " " $\angle F$

Let $\triangle XYZ \sim \triangle PQR$

$\angle X \cong \angle P \mid \angle Y \cong \angle Q \mid \angle Z \cong \angle R$

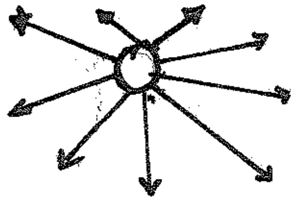




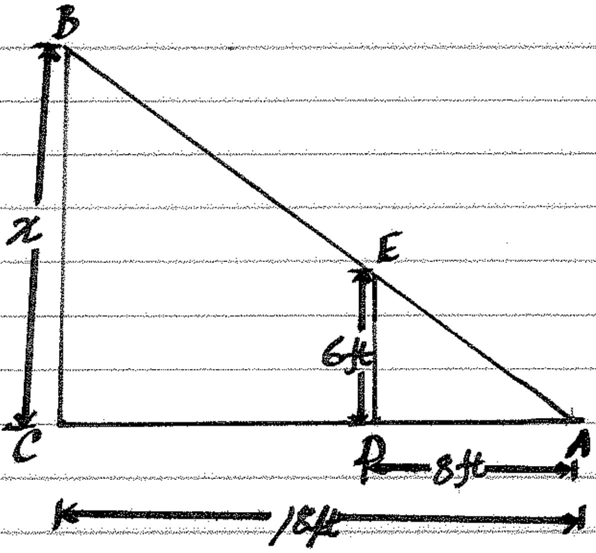
$\Rightarrow$

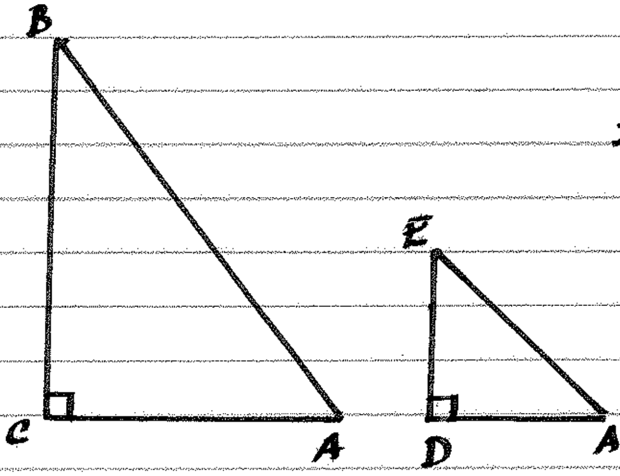
$$\boxed{\frac{\overline{XY}}{PQ} = \frac{\overline{YZ}}{QR} = \frac{\overline{XZ}}{PR}}$$

$\triangle \overline{XYZ} \sim \triangle \overline{PQR}$



A 6ft tall man is standing infront of a tree. If the man casts an 8ft shadow and the tree casts 18ft shadow, how tall is the tree. (Assume that the two shadows ends at the same point)





From the figure, we can create two triangles visuing the tree and the man together with their shadows.

$\triangle AED \sim \triangle ABC$

$\Rightarrow \frac{AE}{AB} = \frac{AD}{AC} = \frac{ED}{EC}$

since $\overline{AE}$ & $\overline{AB}$ are not given dimension, we exclude them from the proportion and use only $\frac{AD}{AC} = \frac{ED}{EC}$

Substituting their values $\frac{AD}{AC} = \frac{ED}{EC}$ $\qquad \frac{8ft}{18ft} = \frac{6ft}{x}$

$$\frac{x(8ft)}{8ft} = \frac{(18\times 6)ft^2}{8ft}$$

$$\boxed{x = 13.5\ ft}$$

The height of the tree is 13.5 ft.

A percent is a portion of something out of a 100.

normally percents are writen followed by '%' symbol

Eg:

20%, 5%, 2.5%, 15%, 250%

notice that a percent could be more than a 100, But we don't see it regularly.

Percents are very useful in comparing unrelated items.

for example two students have \$100 each. If each student spend \$25 to buy a book, then both student spend 25% of their money.

what if the 1^{st} student has only \$50 and still spent \$25 to buy the book?

In this case the 1^{st} student spent 50% of his/her money.

By just looking the money they spent, we can not really compare whom spent the most money based on what they had.

There are three common variable associated with percent.

1. percent (P)
2. percent amount (PA)
3. Total amount (TA)

a percent to a 100 is proportional to PA to TA.

$$\boxed{\frac{P}{100} = \frac{PA}{TA}}$$

Using the above formula, we can answer questions like

1. what is 10% of 60?
2. 6 is what percent of 60?
3. 6 is 10% of what?

The 1st question is asking the percent amount (PA)
The 2nd question is asking the percent (P)
The 3rd question is asking the total amount (TA)

we can use cross product to find all of the above questions. i.e.

$$\frac{P}{100} = \frac{PA}{TA}$$

$$\frac{10}{100} = \frac{x}{60}$$

$$(10)(60) = x(100)$$

$$\frac{600}{100} = \frac{100x}{100}$$

$$\boxed{6 = x}$$

Examples

1. 15 is what percent of 45?

Required: percent (P)

$$\frac{P}{100} = \frac{PA}{TA}$$

$$\frac{x}{100} = \frac{15}{45}$$

$$(45)(x) = (15)(100)$$

$$\frac{45x}{45} = \frac{1500}{45}$$

$$\boxed{x = 33.3\%}$$

2. 12 is 30% of what?

Required: Total amount (TA)

$$\frac{P}{100} = \frac{PA}{TA} \quad ; \quad \frac{30}{100} = \frac{12}{x}$$

$$30x = (12)(100)$$

$$\frac{30x}{30} = \frac{1200}{30}$$

$$\boxed{x = 40}$$

3. What is 150% of 40?

Required: percent amount (PA)

$$\frac{P}{100} = \frac{PA}{TA} \quad ; \quad \frac{150}{100} = \frac{x}{40}$$

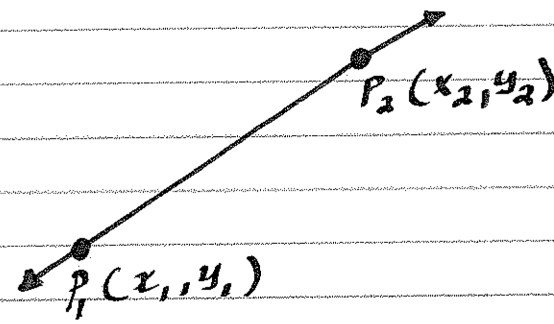
$$(150)(40) = (100)(x)$$

$$\frac{6000}{100} = \frac{100x}{100}$$

$$\boxed{60 = x}$$

Linear Equations

There is only one unique line that passes through two point.



The graph of a linear equation is a line, more specifically a straight line

A point can be represented by a coordinate. The 1st entry of a coordinate is the x-value and the second entry is called the y-value.

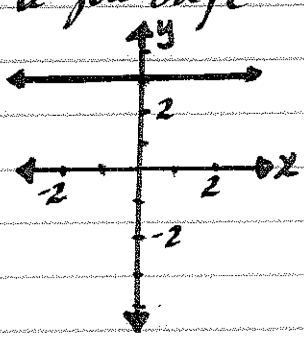
The slope of a line is the rate in which the y value changes with respect to x-value change. In short slope is defined as rise over run.

Rise - the vertical change b/w the y coordinates

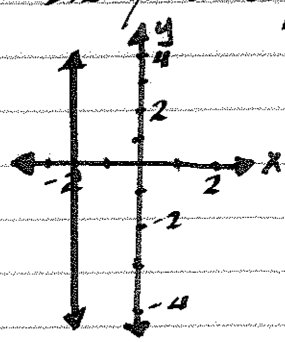
Run - the horizontal change b/w the x coordinates.

There are four different types of slopes.

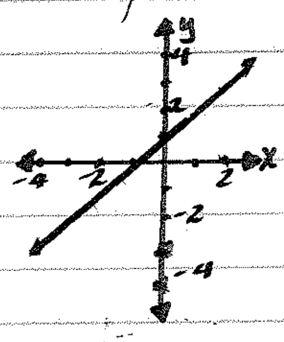
a zero slope



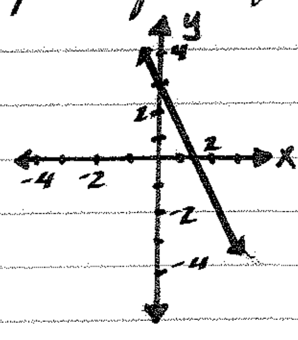
undefined slope



positive slope

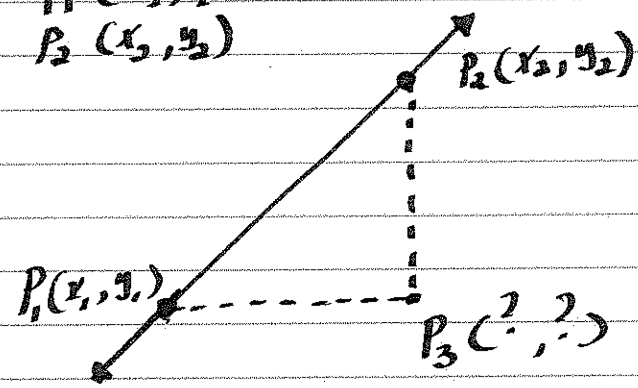


negative slope



Slope

$P_1\ (x_1, y_1)$
$P_2\ (x_2, y_2)$



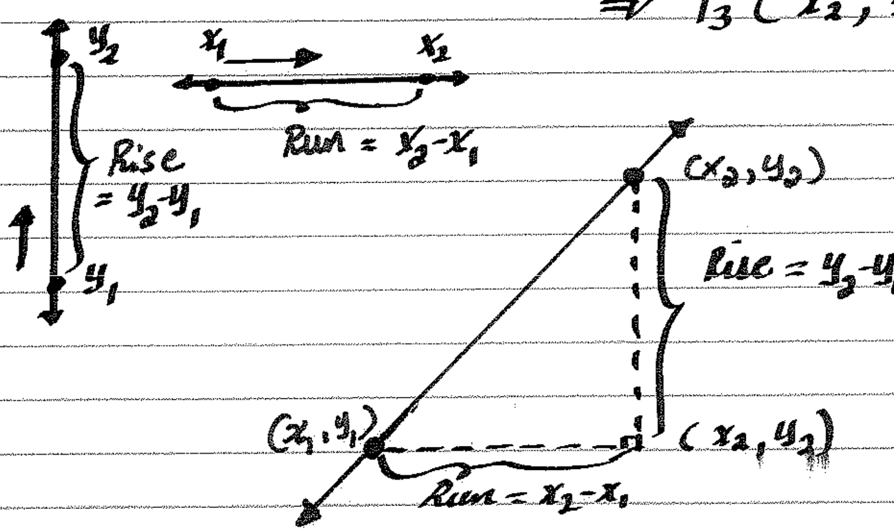
Remark:
The x values remains constant in any vertical lines.

The y values remains constant in any horizontal lines

The coordinates of P_3 are unknow at the moment. But we can find them by applying the coorinates of horifontal and vertical lines properties.

$\Rightarrow$ The y coordinate of P_3 lies on the same horizontal line as P_1
and
The x coordinate of P_3 lies on the same vertical line as P_2.

$\Rightarrow P_3\ (x_2, y_1)$



$$\text{Slope} = \frac{\text{Rise}}{\text{Run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

Slope is represented by the letter 'm'

$$\boxed{m = \frac{y_2 - y_1}{x_2 - x_1}}$$

☆ Notice that the slope of a horizontal line is 0. This is because there is 'no' rise in a horizontal line

$$m = \frac{0}{x_2 - x_1}, \quad x_2 - x_1 \neq 0$$

$$= 0$$

$$m = \frac{y_1 - y_1}{x_2 - x_1} = \frac{0}{x_2 - x_1}$$

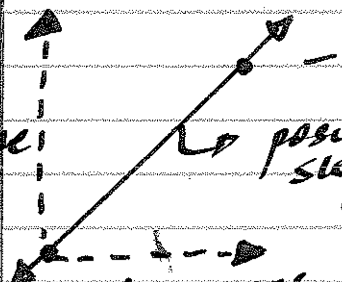
$$m = 0$$

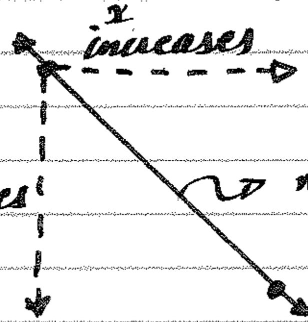
☆ Notice that the slope of a vertical line is undefined. This is because there is no run implying that ~~Run~~ Run = 0

$$m = \frac{y_2 - y_1}{0}$$ Dividing by zero is not possible!

$$m = \frac{y_2 - y_1}{x_2 - x_2} = \frac{y_2 - y_1}{0}$$

$$m = \nexists$$





As the slope of a line gets bigger and bigger, the line gets steeper and steeper

As the slope of a line approaches to zero, the line gets flatter and flatter.

Examples:

Eg1. Find the slope of a line that passes through

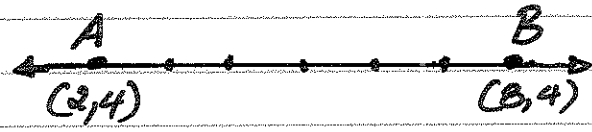
$A(2,4) \quad B(8,4)$

Let $A(2,4) \quad B(8,4)$
$\quad\;\; x_1 \; y_1 \qquad\quad x_2 \; y_2$

$m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$= \dfrac{4-4}{8-2} = \dfrac{0}{6}$

$\underline{\underline{m = 0}}$



Eg2. Find the slope of a line that passes through

$C(-3,5) \quad D(3,6)$

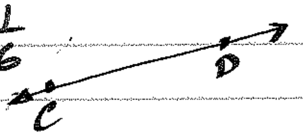
Let $C(-3,5) \quad D(3,6)$
$\quad\;\; x_1 \; y_1 \qquad\quad x_2 \; y_2$

$m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$= \dfrac{6-5}{3-(-3)}$

$= \dfrac{1}{3+3} = \dfrac{1}{6}$

$m = 1/6$



Eg3. Find the slope of a line that passes through

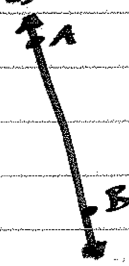
$A(-2,10) \quad B(1,3)$

Let $A(-2,10) \quad B(1,3)$
$\quad\;\; x_1 \; y_1 \qquad x_2 \; y_2$

$m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{3-10}{1-(-2)} =$

$= \dfrac{-7}{3}$

$m = -7/3$



Eg 4:

Find the slope of a line that passes through

$C(3,8) \quad D(3,-2)$

Let $C(3,8) \quad D(3,-2)$
$\quad\;\; x_1 \; y_1 \qquad x_2 \; y_2$

$m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$= \dfrac{-2-8}{3-3}$

$= \dfrac{-10}{0} \quad \nexists$

$m = \nexists$

Equation of a line

There are three different way a line can be written.

1. Standard form
2. point slope form
3. slope intercept form

Let m represents the slope of a line and b represents they y-intercept.

Slope intercept form:

$$y = mx + b$$

Standard form

$$Ax + By = C$$, a, b and c are integer coefficients

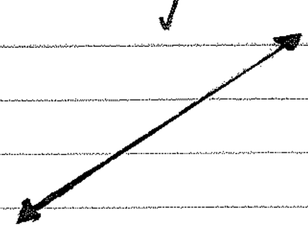
Point slope form

$$y - y_1 = m(x - x_1)$$ m - slope

(x_1, y_1) - a point on the line.

Notice that each form of a line relates x and y.

The graph of a linear equation is a straight line

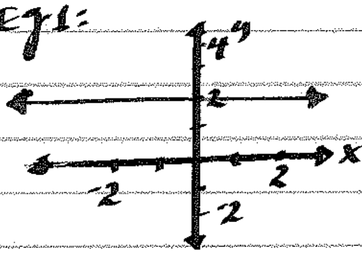


Special Cases:

Horizontal lines. A horizontal line has a zero slope and a constant y value.

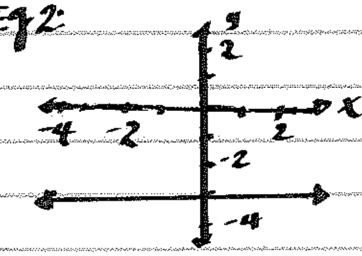
The equation of a horizontal line is always given by the form
$$y = k$$
where k, is the y value, which the line passes through.

Eg 1:



The equation of the line is
$y = 2$, since the line passes through 2 on the y-axis.

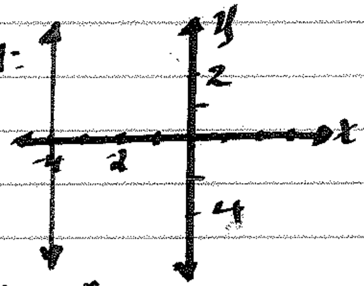
Eg 2:



The equation of the line
$y = k$ since the line passes
$y = -3$ through -3 on the y-axis

Vertical lines A vertical line has undefined slope & a constant x value.

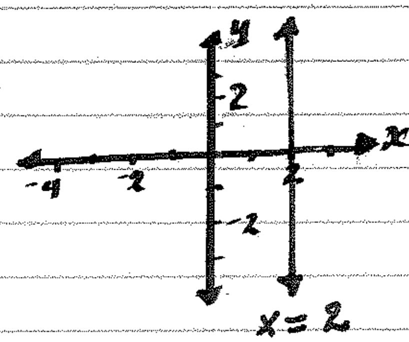
Eg 1:



$x = -4$

The equation of a vertical line is given by the form,
$$x = c$$
where c is the x value, which the line passes through.

Eg 2:



$x = 2$

Techniques on how to write equation of a line:

Depending on the given information, we can write the equation of a line in three forms.

Question:

which equation form can we start?

Answers

If a slope and a point is given: slope-point form

If a slope and y-intercept is given: slope-intercept

If two points are given: point slope form:

Eg 1:

Find the equation of a line that has slope 2 and y-intercept 5.

$y = mx + b$

substitute m & b values

given: $m = 2$

$b = 5$

form: slope-intercept: $\boxed{y = 2x + 5}$

Eg 2:

Find the equation of a line that has slope ½ and passes through the point (-4, 12).

given: $m = \frac{1}{2}$

$(x_1, y_1) = (-4, 12)$

$y - y_1 = m(x - x_1)$

substitute m & (x_1, y_1)

form: point-slope: $\boxed{y - 12 = \frac{1}{2}(x + 4)}$

Eg 3: Find the equation of a line that passes through (3, 1) & (-4, 15)

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{15 - 1}{-4 - 3} = \frac{14}{-7} = -2$

given: $(\underset{x_1}{3}, \underset{y_1}{1})$ & $(\underset{x_2}{-4}, \underset{y_2}{15})$

$y - y_1 = m(x - x_1)$

$\boxed{y - 1 = -2(x - 3)}$ OR

$\boxed{y - 15 = -2(x + 4)}$

first find the slope

form: point-slope:

THREE FORMS of a LINE:

In this section we will learn how to write an equation of a line in all the three forms.

Steps:

1st: Determine which form to start the equation of a line depending on the given information.

2nd: Simplify the 1st equation to the desired form

3rd: Make sure that standard forms of a linear equation has integer coefficients.

Eg: Write the equation of a line that passes through (4,5) & (2,-6) in all the three forms.

1st: let's find the slope:

given: (4,5) & (2,-6)

label → x_1 y_1 x_2 y_2

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-6-5}{2-4} = \frac{-11}{-2} = \frac{11}{2}$$

$m = 11/2$

Now let's use the slope we calculated and the 1st point

form: point-slope form:

$$y - y_1 = m(x - x_1)$$

$$\boxed{y - 5 = 11/2\,(x-4)}$$

To find the slope intercept form:

$$y - 5 = \frac{11}{2}(x-4)$$

$$y - 5 = \frac{11}{2}x - \frac{44}{2}$$

$$y - 5 = \frac{11}{2}x - 22$$

$+5 \qquad +5$

$$\boxed{y = \frac{11}{2}x - 17}$$

To find the standard form:

$$y = \frac{11}{2}x - 17$$

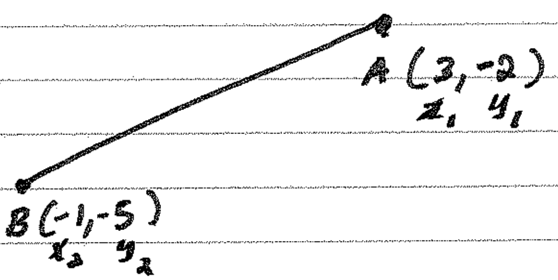
$-\frac{11}{2}x \qquad -\frac{11}{2}x$

$$-\frac{11}{2}x + y = -17$$

$*2 \qquad *2 \qquad *2$

$$\boxed{-11x + 2y = -34}$$

Eg. Find the slope of a line that passes through points $A(3,-2)$ & $B(-1,-5)$



$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{-5-(-2)}{-1-3}$$

$$= \frac{-5+2}{-4}$$

$$= \frac{-3}{-4}$$

$$\boxed{m = 3/4}$$

Eg. Write the equation of a line that has the same slope as the previous example $(3/4)$ and has y-int 5.

Given:

y-int $= 5 = b$

$m = 3/4$

Lets start with slope intercept form: $y = mx + b$

$$\boxed{y = \tfrac{3}{4}x + 5}$$

standard form:

$$y = \frac{3}{4}x + 5$$

$$-\tfrac{3}{4}x \quad -\tfrac{3}{4}x$$

$$-\frac{3}{4}x + y = 5$$

$$*4 \quad *4 \quad *4$$

$$\boxed{-3x + 4y = 20}$$

point slope form - we have slope $m = 3/4$ but we need to find a point x_1, y_1

notice: $P(4, 8)$ is on the line:

$$y - y_1 = m(x - x_1)$$

$$\boxed{y - 8 = \frac{3}{4}(x - 4)}$$

Eg: write the equation of a line with slope $-1/4$ and has a point $(2,6)$; in all the three forms:

Given:

$m = -1/4$

$p = (\underset{x_1}{2}, \underset{y_1}{6})$

Start with:

point-slope form:

$$y - y_1 = m(x - x_1)$$

$$\boxed{y - 6 = -1/4(x-2)}$$

To find slope-intercept distribute $-1/4$ over $(x-2)$ & simplify.

$$y - 6 = \frac{-1}{4}(x-2)$$

$$y - 6 = -1/4x + \frac{1}{4}(2), \quad \frac{1}{4}\cdot\frac{2}{1} = \frac{2}{4} = 1/2$$

$$y - 6 = -1/4x + 1/2$$

$$+6 \qquad\qquad +6$$

$$1/2 + \frac{6}{1}$$

$$\frac{1}{2} + \frac{12}{2}$$

$$\frac{13}{2}$$

$$\boxed{y = -1/4x + \frac{13}{2}}$$

To find standard form: subtract $(-1/4x)$ from both sides & multiply everything by 4.

$$y = -1/4x + \frac{13}{2}$$

$$+1/4x \qquad +1/4x$$

$$\frac{1}{4}x + y = \frac{13}{2}$$

$$*4 \qquad *4 \qquad *4$$

$$\boxed{x + 4y = 26}$$

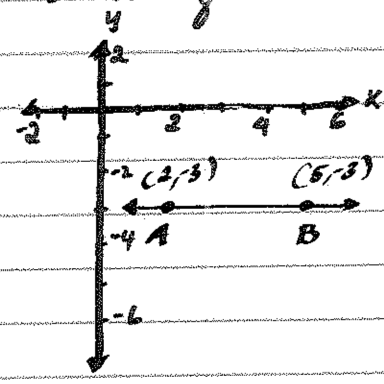
☆ Notice that once we find point-slope form of line, its a matter of simplifying the equation to get the other forms.

Horizontal & Vertical equations of a line

Recall from earlier Section: The equation of a horizontal & a vertical line is given by
$y = k$ & $x = a$ respectively
where k & a are constants.

How can we tell if a line that passes through two points is a horizontal or a vertical line? $\Longrightarrow$ If the points have the same x-coordinate, then the line is a vertical line. If the two points have the same y-coordinate, then the line is a horizontal line.

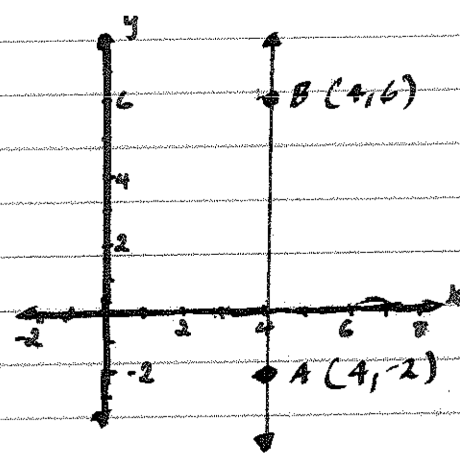
Eg: Find the equation of a line that passes through points $A(2,-3)$ & $B(5,-3)$



Notice that both points have the same y-coordinate. This implies that the line is a horizontal line

$\boxed{y = -3}$

Eg. Find the equation of a line that passes through points $A(4,-2)$ & $B(4,6)$

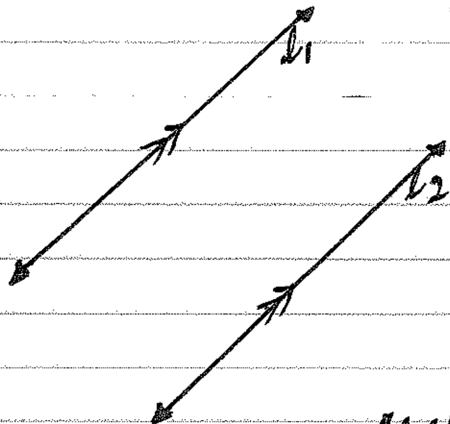


Notice that both points have the same x-coordinates. This implies that the line is vertical

$\boxed{x = 4}$

Parallel And Perpendicular lines

Parallel lines are lines on a given plane that do not intersect at any point



Symbol (parallel "$\parallel$")
In the above figure
l_1 is parallel to l_2
$\Rightarrow l_1 \parallel l_2$

If two lines are parallel then they must have the same "slope".

Remark: parallel lines can not have the same slope and the same y int.
If this happens, the lines are exactly the same.

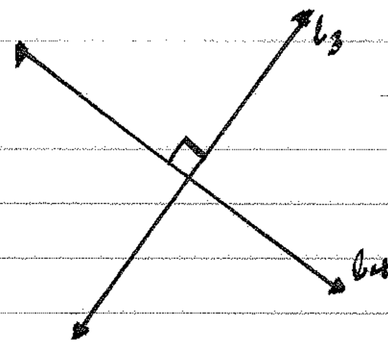
Eg: l_1: $y = 7x + 2$
l_2: $y = 7x - 5$

let m_1 be the slope of l_1 & m_2 be the slope of l_2

$m_1 = 7$ since the y-int are
$m_2 = 7$ different and $m_1 = m_2$

$\boxed{l_1 \parallel l_2}$

Perpendicular lines are lines on a given plane that intersect at 90°.



Symbol (perpendicular ("$\perp$"))
In the above figure
l_3 is perpendicular l_4
$\Rightarrow l_3 \perp l_4$

If two lines are perpendicular, then the "product" of their "slope" must be "-1"

Eg: l_3: $y = 2x - 6$
l_4: $y = -\frac{1}{2}x + 3$

let m_1 be slope of l_3 & m_2 be slope of l_4.

$m_1 = 2$
$m_2 = -\frac{1}{2}$

now the product of m_1 & m_2 is
$m_1 * m_2$
$2 * -\frac{1}{2}$
$\boxed{= -1}$ $\Rightarrow$ $\boxed{l_3 \perp l_4}$

Eg. Find an equation of a line that is parallel to $y = 3x - 2$ and passes through $(-2,3)$

Given: Eq 1: $y = 3x - 2$

$P(-2, 3)$

parallel lines have the same slope.

$\Rightarrow m = 3$

Since we have slope & a point, we can use point slope form of a line to find the equation.

$$y - y_1 = m(x - x_1)$$

$m = 3$

$P(\underset{x_1}{-2}, \underset{y_1}{3})$

$$y - 3 = 3(x - (-2))$$

$$y - 3 = 3x + 2$$

$$+3 \qquad +3$$

$$\boxed{y = 3x + 5}$$

Eg. Find the equation of a line that is perpendicular to the line $2x + 3y = 10$ and passes through the point $(4,1)$.

given:

Eq 1: $2x + 3y = 10$

$P(4, 1)$

we can find the slope of our line from the given equation by using properties of perpendicular lines.

i.e the product of their slope is -1.

$2x + 3y = 10$ needs to be ~~..~~ rewritten

$-2x \qquad -2x$ in $y = mx + b$ form

$$\frac{3y}{3} = \frac{-2x}{3} + \frac{10}{3}$$

$$y = -\frac{2}{3}x + \frac{10}{3}$$

Since the slope $= -\frac{2}{3}$, our slope is $\frac{3}{2}$.

Check, $-\frac{2}{3} \times \frac{3}{2} = -1$.

now we have $m = \frac{3}{2}$, $P(4,1)$

$$y - y_1 = m(x - x_1)$$

$$y - 1 = \frac{3}{2}(x - 4)$$

$$y - 1 = \frac{3}{2}x - 6$$

$$+1 \qquad +1$$

$$\boxed{y = \frac{3}{2}x - 5}$$

Eg:
Find an equation of a line that is perpendicular to the line $2x + y = 4$ and has the same y-intercept as the line $4x + 2y = -6$

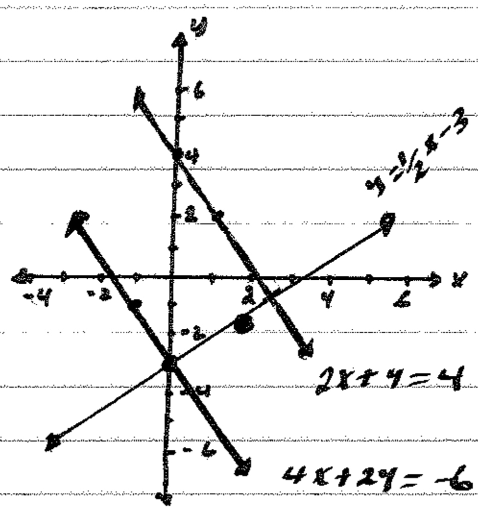
Given: two equations:

Eq 1. $2x + y = 4$ (perpendicular to this line)

Eq 2: $4x + 2y = -6$ (has the same y-int as this line)

Strategy.

* get the slope of our line from the 1^{st} eq and get the y-int of our line from the second equation



1^{st} we need to rewrite both equations in slope intercept form.

Eq 1:

$2x + y = 4$

$-2x \qquad -2x$

$y = -2x + 4$

Eq 2:

$4x + 2y = -6$

$-4x \qquad -4x$

$\frac{2y}{2} = \frac{-4x}{2} - \frac{6}{2}$

$y = -2x - 3$

$\Rightarrow$ our slope is $\frac{1}{2}$ since $-2 * \frac{1}{2} = -1$.

$\Rightarrow$ our y-int is -3 since the are the same.

Our equation $y = mx + b$ $\quad m = \frac{1}{2}$, $b = -3$

$$\boxed{y = \frac{1}{2}x - 3}$$

Systems of Linear Equations:

A system is when a couple groups of things get combined to create something.

In this specific case:

* System of linear equations is defined as <u>two</u> linear equations combined together.

Remark: There could be more than two linear equation, but for now we are only going to study two linear equations.

Notation:

we use a curly bracket to represent a system of linear equations. " $\{$ "

Eg: $\begin{cases} y = 2x + 1 \\ y = -x + 10 \end{cases}$ $\begin{cases} 3x - 2y = 5 \\ 2x + 1y = 4 \end{cases}$ $\begin{cases} \frac{1}{2}x - 3y = 6 \\ y = 2x + 2 \end{cases}$

Notice that each system of linear equations can be written as slope-intercept form, standard form or in both forms.

Solving a systems of linear equations is basically find values of "x" & "y" that makes both linear equations true.

For example $x = 1$ $y = 5$ $(1,5)$ is a solution for the system $\begin{cases} y = 2x + 3 \\ y = -3x + 8 \end{cases}$

" This is because if you put both $x = 1$ and $y = 5$ in each equation, you will get a true statement. "

Depending on the linear equations, a system of linear equation can have one of the following three solution types.

1. One solution
2. Many solutions
3. No solution

<u>One solution</u>

This type exist if and only if there is exactly "one" solution that makes both linear equations true.

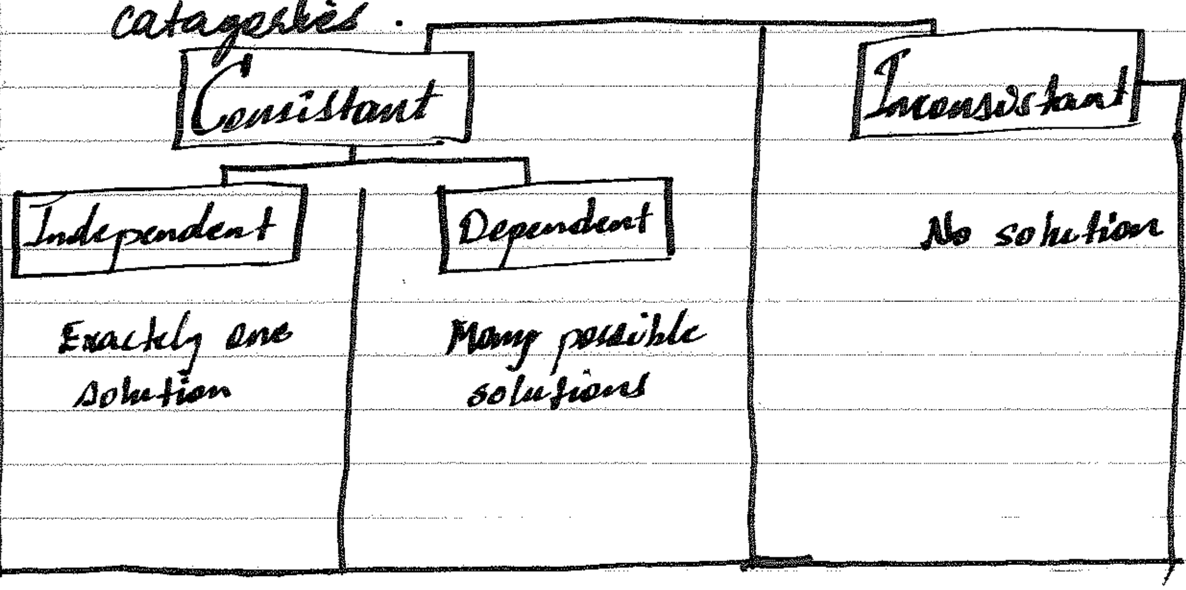
<u>Many solutions</u>

This type exists if there are many x and y that makes both linear equations true

<u>No solutions</u>

This type exists, if there are no such x & y that makes both equations true.

Based on the solution type we can classify systems of linear equations in ~~to~~ two catagories.



Systems of Linear Equation
Solving Techniques

There are three different techniques to solve systems of linear equations.

I	II	III
Graphing	Substitution	Elimination
we can draw the graph of the two given equation and find the point where the two lines intersect.	we can rewrite one of the linear equation in slope intercept form and substitute 'y' in the other equation to solve for 'x' and then 'y'	we can write both equation in standard form and eliminate either the x or y value & solve for the other.

$\boxed{\text{These techniques make much more sense through examples.}}$

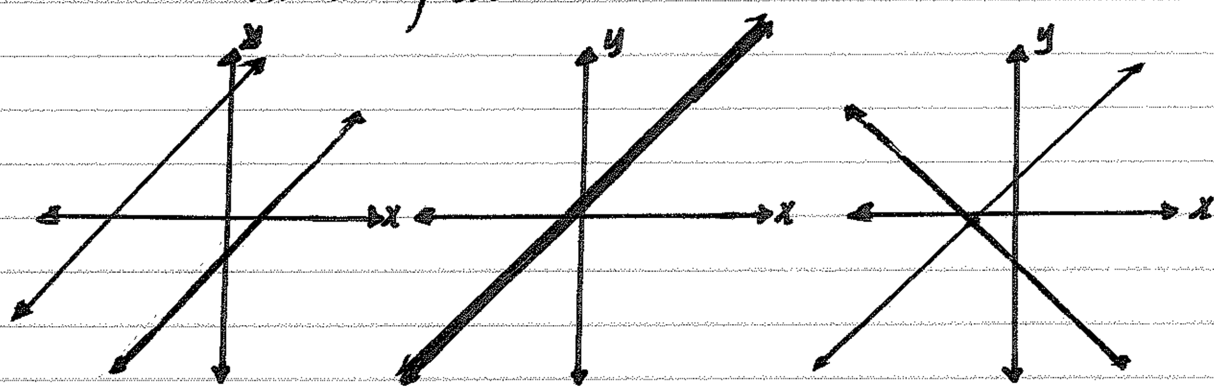
Eg. which method would be more convenient to solve the following system of linear equations.

1. $\begin{cases} 2x - y = 8 \\ 2x + 4y = 6 \end{cases}$ "Elimination"

2. $\begin{cases} 2x + 3x = 12 \\ 3x + y = 6 \end{cases}$ "Graphing"

3. $\begin{cases} y = 2x + 1 \\ 2x + y = 4 \end{cases}$ "Substitution"

How to determine whether
a given system of linear equations
is consistant, inconsistant, dependent
or independent.



parallel lines	Same lines	Intersecting lines
"Inconsistant"	"Consistant" Dependent	"Consistant" Independent

If the two lines have the same slope but different y-intercepts, then the lines are parallel and hence not intersecting.

If the two lines have the same slope and identical y-intercepts, then the lines are exactly the same and they intersect everywhere.

consistant and independent solutions exist when the slope of the two equation are different.

Classify the following system of linear equations as consistant, inconsistant, dependend and independent.

Eg 1.
$$\begin{cases} y = 3x + 5 \\ y = 2x - 1 \end{cases}$$

the slope of the 1st eq. is 3 and the 2nd eq. is 2. This implies that the system is <u>consistant and independent</u>

Eg. 2
$$\begin{cases} y = 3x - 5 \\ y = 3x + 2 \end{cases}$$

the slope of the 1st eq is 3 and the 2nd eq is 3. since both equations have the same slope and different y-intercepts, the lines are parallel implying the system is <u>inconsistant</u>.

Eg 3.
$$\begin{cases} y = \frac{1}{2}x + 3 \\ -6x + 12y = 36 \end{cases}$$

1st change the second equation into slope-intercept form

$$-6x + 12y = 36$$
$$+6x \qquad\quad +6x$$
$$\frac{12y}{12} = \frac{6}{12}x + \frac{36}{12}$$
$$y = \frac{1}{2}x + 3$$

Both equations have the same slope, i.e ($\frac{1}{2}$) and y-intercept i.e (3), the equations are exactly identical.

This implies, the system is <u>consistant</u> and <u>dependent</u>.

Eg.

Solve the following system of linear equation by graphing.

$$\begin{cases} 2x + 3y = 12 \\ 3x + y = 6 \end{cases}$$

Since both equations are given in standard form, we can easily find the x and y intercepts

Eq 1. $2x + 3y = 12$

x-int, plug $y = 0$

$2x + 3(0) = 12$

$\frac{2x}{2} = \frac{12}{2}$

$x = 6$

y-int, plug $x = 0$

$2(0) + 3y = 12$

$\frac{3y}{3} = \frac{12}{3}$

$y = 4$

x-int $(6, 0)$

y-int $(0, 4)$

Eq 2. $3x + y = 6$

x-int, plug $y = 0$

$3x + (0) = 6$

$\frac{3x}{3} = \frac{6}{3}$

$x = 2$

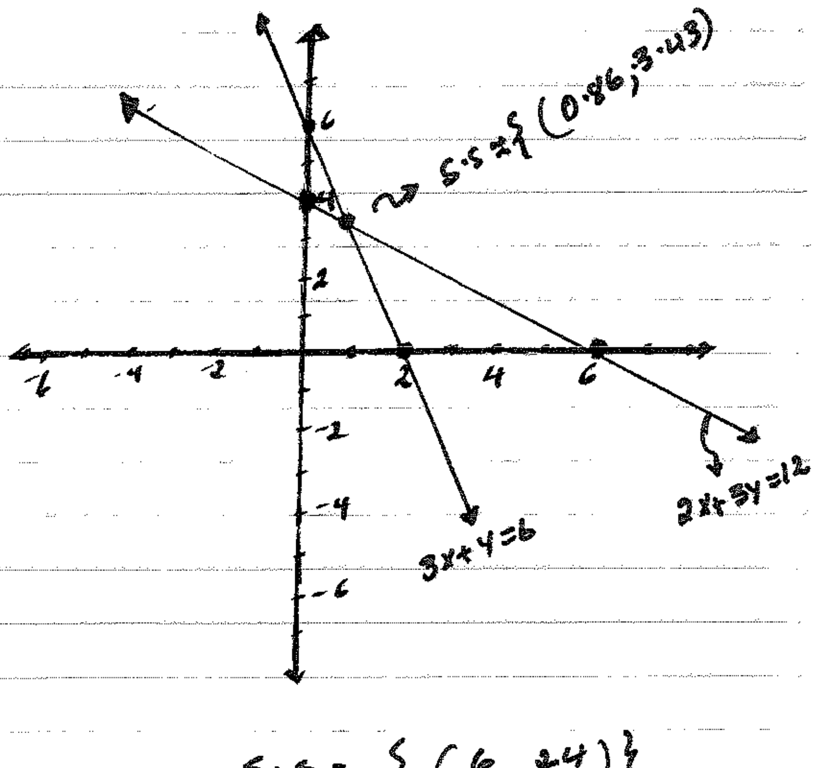
y-int, plug $x = 0$

$3(0) + y = 6$

$y = 6$

x-int $(2, 0)$

y-int $(0, 6)$



$S.S = \{(\frac{6}{7}, \frac{24}{7})\}$

Eg.
Solve the following system of linear equations by Substitution.

$$\begin{cases} y = 2x+1 \\ 2x+y=4 \end{cases}$$

Since the first equation is given in slope-intercept form, we can substitute it in the 2nd equation

$y = 2x+1$

Substitute the 'y' in the second equation by $2x+1$

$2x + y = 4$

$2x + (2x+1) = 4$

$2x + 2x + 1 = 4$

$4x + 1 = 4$

$-1 = -1$

$\frac{4x}{4} = \frac{3}{4}$

$\boxed{x = 3/4}$

Now: since we know the value of 'x' we can put the 'x' value in the 1st equation to find 'y'.

$y = 2x+1$

$y = 2\left(\frac{3}{4}\right)+1$

$y = \frac{6}{4}+1$

$y = \frac{6}{4}+\frac{4}{4}$

$y = \frac{6+4}{4} = \frac{10}{4}$

$\boxed{y = \frac{5}{2}}$

$S.S. = \{(3/4, 5/2)\}$

Eg. Solve the following by elimination

$$\begin{cases} 2x - y = 8 \\ 2x+4y = 6 \end{cases}$$

Subtract the 2nd equation from 1st

$$-\begin{cases} 2x - y = 8 \\ 2x+4y = 6 \end{cases}$$

$-y - 4y = 8 - 6$

$\frac{-5y}{-5} = \frac{2}{-5}$

$\boxed{y = \frac{-2}{5}}$

plug y in the 1st eq.

$2x - \left(\frac{-2}{5}\right) = 8$

$2x + \frac{2}{5} = 8$

$2x = 8 - \frac{2}{5}$

$\frac{2x}{2} = \frac{\frac{38}{5}}{2}$ $\boxed{x = \frac{19}{5}}$

$S.S = \{(\frac{19}{5}, \frac{-2}{5})\}$

Example:

Determine whether the following system has a consistant or inconsistance solution and find the solution by using all the three techniques i.e. graphing, substitution & Elimination

$$\begin{cases} y = 2x + 2 \\ 2x + 3y = 6 \end{cases}$$

Let m_1 be slope of eq 1 and m_2 be eq 2.

$\boxed{m_1 = 2}$, to find the m_2, we need to rewrite the 2nd eq in slope-int form:

$$2x + 3y = 6$$
$$-2x \qquad -2x$$
$$\frac{3y}{3} = \frac{-2x}{3} + \frac{6}{3}$$
$$y = -\frac{2}{3}x + 2$$
$$\Rightarrow \boxed{m_2 = -\frac{2}{3}}$$

Since the two equations have different slopes, the system is <u>consistant</u> and <u>independent</u>.

a) substitution:

$$\begin{cases} y = 2x + 2 \\ 2x + 3y = 6 \end{cases}$$

$$2x + 3(2x+2) = 6$$
$$2x + 6x + 6 = 6$$
$$8x + 6 = 6$$
$$-6 \quad -6$$
$$8x = 0$$
$$x = 0$$

$$y = 2x + 2$$
$$y = 2(0) + 2$$
$$y = 2$$

$$S.S = \{(0, 2)\}$$

b) Elimination

$$y = 2x + 2$$
$$-2x \quad -2x$$
$$\begin{cases} -2x + y = 2 \\ 2x + 3y = 6 \end{cases}$$
$$+ \quad \overline{\frac{4y}{4} = \frac{8}{4}}$$
$$y = 2$$
$$y = 2x + 2$$
$$2 = 2x + 2$$
$$-2 \qquad -2$$
$$2x = 0 \quad x = 0$$

c) Graphing

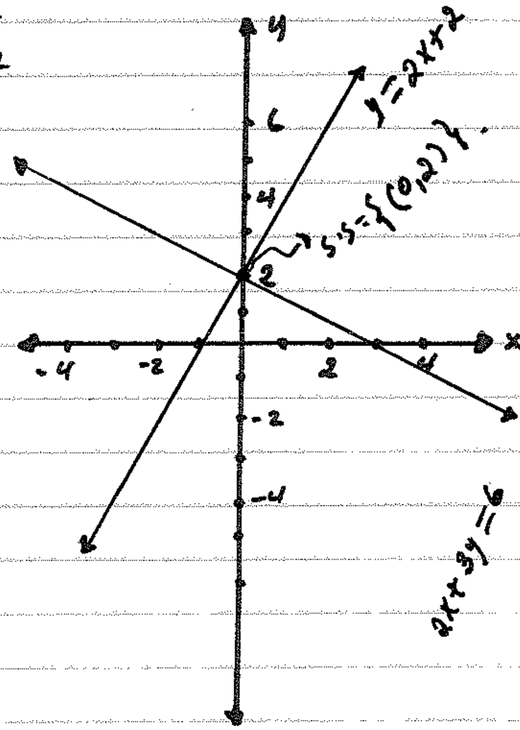
$y = 2x + 2$

x	y
0	2
-1	0

$2x + 3y = 6$

x	y
0	2
3	0

$$S.S = \{(0, 2)\}$$



Linear Inequalities

A linear inequality is an inequality with a linear function

From the definition a linear inequality has three components

1. has to be a function
2. has to be a linear
3. needs to have an inequality sign

Eg:

a) $y < 3x - 1$

b) $5x - 2y \geq 5$

c) $x > 4$

These are all linear inequalities

A solution of a linear inequality is a set of points (x, y) that make the inequality true.

Eg.

a) $y < 3x - 1$ $\quad$ $\overset{x}{(4},\overset{y}{2)}$ is a solution because

$2 < 3(4) - 1$

$2 < 12 - 1$

$2 < 11$ true

Unlike linear equations, linear inequalities have more than one solution

In the above example $\overset{x}{(5},\overset{y}{1)}$ is also a solution, because

$1 < 3(5) - 1$

$1 < 15 - 1$

$1 < 14$ true.

In order to find all the solutions of a given linear inequality, we need to draw its graph and shade the region of the solution set on the coordinate plane.

Technique for graphing and shading a linear inequality

1st. Make sure whether the line is broken or solid.

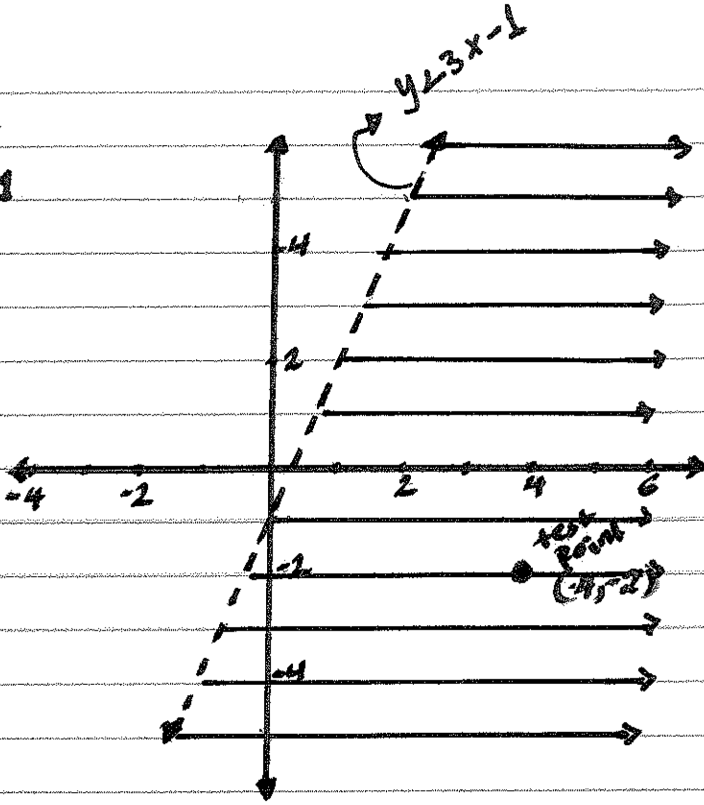
If the inequality sign is	$<$ or $>$	Broken line -------
	$\leq$ or $\geq$	Solid line ———

Broken line means, the solution does not include all the points on the line and solid lines means, the solution set includes the points on the line too.

2nd Draw the graph of the inequality. Find two points by changing the inequality into an equation but don't forget step 1 when you draw the line.

3rd Any line divides the coordinate plane into two parts. we just need to shade one of the two parts. In order to decide which way/part to shade, choose a $\boxed{\text{test point}}$. Plug the test point into the inequality and see if it is true. If it is true shade the graph part that includes the test point. If not true shade the other part.

Example:
Find the solution set by graphing for $y < 3x - 1$



1^{st} step:
Broken or solid line?
Since the inequality sign is '$<$', the line is $\boxed{\text{broken}}$

2^{nd} step:
Change the inequality in to an equation and find two points.

$y = 3x - 1$

x	y
0	-1
2	5

when $x = 0$
$y = 3(0) - 1$
$y = -1$

when $x = 2$
$y = 3(2) - 1$
$= 6 - 1$
$y = 5$

Recall that we can choose any 'x' values but the 'y' values depends on the 'x' values we choose

Now we can draw the broken line through $(0,-1)$ & $(2,5)$

3^{rd} step:
Find any test point and check whether the inequality at that point is true or false.

Test point $(\underset{x}{4}, \underset{y}{-2})$

Any point inside the shaded area/region is a solution

$y < 3x - 1$
$-2 < 3(4) - 1$
$-2 < 12 - 1$
$-2 < 11$ True

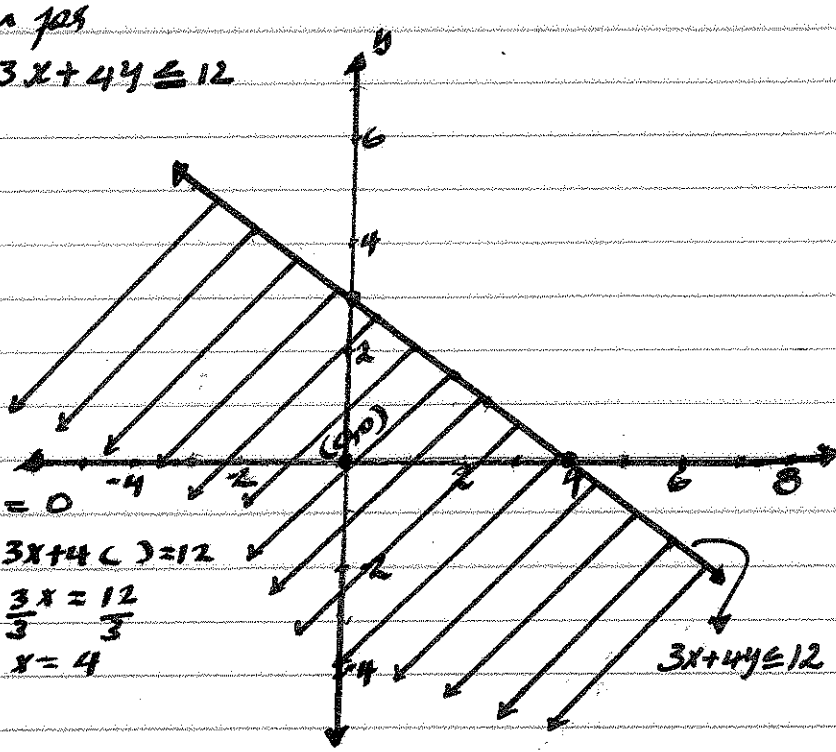
Therefore; we shade the graph towards the test point starting at the line.

Eg: Find the solution for the inequality $3x + 4y \le 12$

1^{st} step:

Inequality sign $\le$

$\boxed{\text{a solid line}}$



2^{nd} step:

points

x	y
0	3
4	0

$3x + 4y = 12$

$x = 0$

$3(0) + 4y = 12$

$\frac{4y}{4} = \frac{12}{4}$

$y = 3$

$y = 0$

$3x + 4(\ ) = 12$

$\frac{3x}{3} = \frac{12}{3}$

$x = 4$

3^{rd} step:

Test point $= (0,0)$

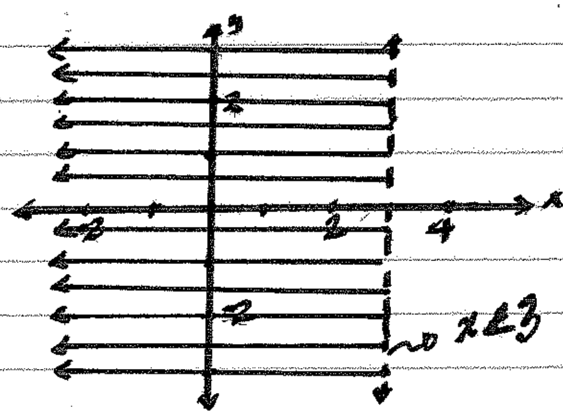
$3x + 4y \le 12$

$3(0) + 4(0) \le 12$

$0 \le 12$ True $\Rightarrow$ shade towards $(0,0)$

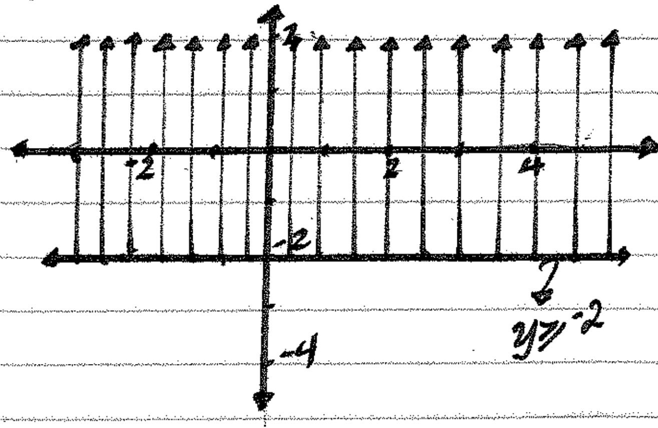
Eg: solution for $x < 3$

It is a vertical line that passes through $x=3$ and it has to be broken line



Eg: solution for $y \ge -2$

It is a horizontal line that passes through $y = -2$ and it needs to be a solid line.



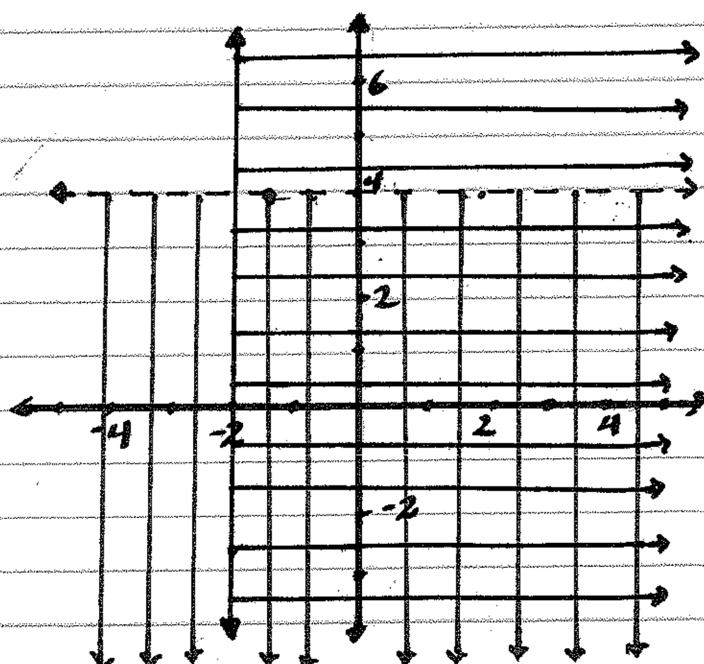
Systems of Linear Inequalities.

A system of linear inequaltis a combination of two or more inequalities combined together!

In order to find a solution for a given system of linear inequalities, we follow the same procedures. just like linear inequalities.

Graphically, a solution of system of linear inequaties is the <u>intersection</u> the individual shaded region.

Eg: $\begin{cases} x \geq -2 \\ y < 4 \end{cases}$



$x \geq -2$

a solid vertical line that passes through $x = -2$

Shaded to the right of $x = -2$

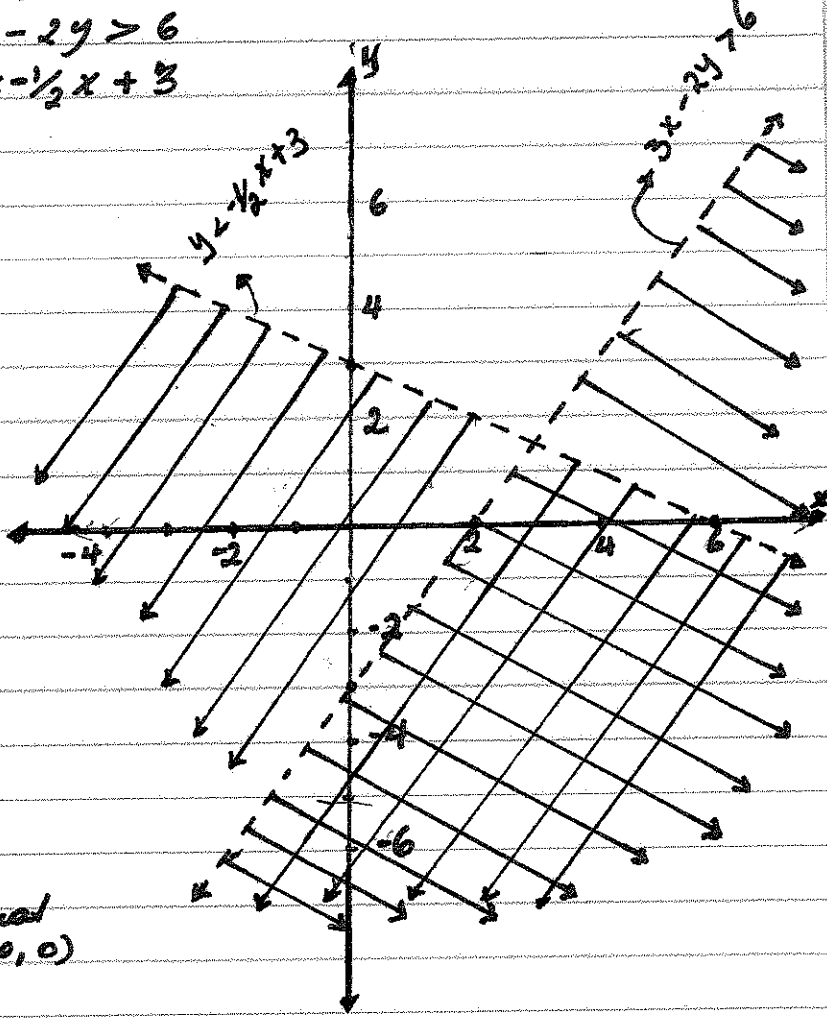
$y < 4$

a broken horizontal line that passes through $y = 4$.

shaded down from $y = 4$

The squared region is the intersection region as a result, it is the solution

Eg:
Solve the system of linear inequalities $\begin{cases} 3x - 2y > 6 \\ y < -\frac{1}{2}x + 3 \end{cases}$



$\boxed{3x - 2y > 6}$

1st step:
'$>$' broken line

2nd step:
$3x - 2y \geq 6$

$x = 0$ $\quad\quad$ $y = 0$

$3(0) - 2y = 6$ $\quad\quad$ $3x - 2(0) = 6$

$\frac{-2y}{-2} = \frac{6}{-2}$ $\quad\quad$ $\frac{3x}{3} = \frac{6}{3}$

$y = -3$ $\quad\quad$ $x = 2$

3rd step:
Test point $(0,0)$

$3x - 2y > 6$

$3(\ ) - 2(\ ) > 6$

$0 > 6$ False

shade the region that doesn't include $(0,0)$

$\boxed{y < -\frac{1}{2}x + 3}$

1st step:
'$<$' broken line

2nd step:
$y = \frac{1}{2}x + 3$

$x = 0$ $\quad\quad$ $x = 6$

$y = -\frac{1}{2}(0) + 3$ $\quad\quad$ $y = -\frac{1}{2}(6) + 3$

$y = 3$ $\quad\quad$ $= -3 + 3$

$y = 0$

points $(0,3)$ and $(6,0)$

3rd step:
Test point $(0,0)$

$y < -\frac{1}{2}(x) + 3$

$0 < -\frac{1}{2}(0) + 3$

$0 < 3$ True

shade towards $(0,0)$

Exponential Function

Functions are formulas that relates two or more things.
Some functions are a bit more complex than others.

Exponential function are a type of functions that correlates "two" variables in a special way.

Before we get to exponential functions, it is important to go over powers.

Exponent.

$$y = a(b)^x$$

Base

Coefficient

Exponent: The exponent tells us how many times we need to multiply the base by itself.

Base: The base gets to be multiplied by itself as many times as the exponent.

Coefficient: The coefficient multiplies the final out come of the base and the exponent.

Examples:

a) 4^3

$= \underbrace{4 * 4 * 4}_{3}$

$= 64$

b) $2(3)^4$

$= 2(3 * 3 * 3 * 3)$

$= 2(81)$

$= 162$

c) $-3(2)^5$

$-3(2 * 2 * 2 * 2 * 2)$

$-3(32)$

-96

Fundamental Properties of exponents:

There are 5 fundamental properties of exponents.

1. $x^0 = 1$ Anything the power of '0' is 1. Exception would be $0^0 \neq \mathbb{Z}$

2. $x^{-n} = \frac{1}{x^n}$ We can always convert a negative exponent into positive by making it a reciprocal.

3. $x^a * y^a = (xy)^a$ OR $x^a / y^a = (x/y)^a$ If two powers have the same exponents and they are connected by either multiplication or division, we can combine them under one exponent accordingly.

4. $x^a * x^b = x^{(a+b)}$, $x^a / x^b = x^{(a-b)}$ If two powers have the same base and connected by multiplication, we can combine them by adding the exponents and if they are connected by division we can subtract their exponents.

5. $(x^m)^n = x^{(mn)}$

If we have two exponents for one base, then we can combine the two exponents by multiplication!

Fundamental properties of exponents summary:

1.	$x^0 = 1$
2.	$x^{-n} = 1/x^n$
3.	$x^a * y^a = (xy)^a$ $x^a / y^a = (x/y)^a$
4.	$x^a * x^b = x^{(a+b)}$ $x^a / x^b = x^{(a-b)}$
5.	$(x^m)^n = x^{(mn)}$

Examples:

a) simplify the following

$$-3x^{-2} = -3\left(\frac{1}{x^2}\right) = \boxed{\frac{-3}{x^2}}$$

b) $4x^4y^2x$

$$= 4(x^4 * x)y^2$$
$$= 4(x^{4+1})y^2$$
$$= \boxed{4x^5y^2}$$

c) $\dfrac{2x^3y^{-2}}{x^2y} = 2\left(\dfrac{x^3}{x^2}\right)\left(\dfrac{y^{-2}}{y^1}\right)$

$$= 2(x^{3-2})(y^{-2-1})$$
$$= 2x^1y^{-3}$$
$$= 2x\left(\frac{1}{y^3}\right)$$
$$= \boxed{\frac{2x}{y^3}}$$

d) $\dfrac{3x^3y^{-3}z^3}{x^{-2}z}$

$$= 3\left(\frac{x^3}{x^{-2}}\right)\left(\frac{z^3}{z}\right)\left(\frac{1}{y^3}\right)$$
$$= 3x^{3-(-2)}z^{(3-1)} \cdot \frac{1}{y^3}$$
$$= 3x^5\frac{1}{y^3} \cdot z^2$$
$$= \boxed{\frac{3x^5z^2}{y^3}}$$

e) $(3x^2y)^2 * \dfrac{(y^2)^2}{3}$

$$= (3)^2(x^2)^2(y)^2 \cdot \frac{(y^2)^2}{3}$$
$$= \frac{9x^4y^2 \cdot y^4}{3}$$
$$= \left(\frac{9}{3}\right)x^4(y^{2+4})$$
$$= \boxed{3x^4y^6}$$

More examples:

Simplify the following

Eg 1:

$(-2x^{-2})^3 x^6$

$(-2)^3 (x^{-2})^3 (x^6)$

$-8x^{-6}x^6$

$-8x^{(-6+6)}$

$-8x^0$

$-8(1)$

$\boxed{-8}$

Eg 2:

$\dfrac{4x^{-2}p^3q^{-3}y^2}{(2x)^2 q^{-2}}$

$\dfrac{4x^{-2}q^{-3}p^3y^2}{4x^2q^{-2}}$

$\left(\dfrac{4}{4}\right)\left(\dfrac{x^{-2}}{x^2}\right)\left(\dfrac{q^{-3}}{q^{-2}}\right)\left(p^3y^2\right)$

$1(x^{-2-2})(q^{-3-(-2)})\ p^3y^2$

$x^{-4}q^{-1}p^3y^2$

$\left(\frac{1}{x^4}\right)\left(\frac{1}{q^1}\right)p^3y^2$

$\boxed{\dfrac{p^3y^2}{x^4q}}$

Eg 3:

$\dfrac{(2x^2y)^{-3}}{3x} * \dfrac{y^3}{x^2}$

$\dfrac{(2)^{-3}(x^2)^{-3}(y)^{-3}(y)^3}{3x \cdot x^2}$

$\dfrac{\frac{1}{8}(x^{-6})y^{-3}y^3}{3x^3}$

$\dfrac{1}{(8*3)}(x^{-6-3})(y^{-3+3})$

$\frac{1}{24}x^{-9}y^0$

$\boxed{\dfrac{1}{24x^9}}$

Eg 4:

$\dfrac{(-2x^{-2}y)^2p^4}{x} \div \dfrac{(4xy^{-1})^{-2}}{p^{-3}}$

1st we need to change the division into multiplication by replacing the divisor by its reciprocal.

$\dfrac{(-2x^{-2}y)^2p^4}{x} * \dfrac{p^{-3}}{(4xy^{-1})^{-2}}$

$\dfrac{(-2)^2(x^{-2})^2y^2p^4}{x} * \dfrac{p^{-3}}{(4)^{-2}(x)^{-2}(y^{-1})^{-2}}$

$\dfrac{4x^{-4}y^2p^4}{x} * \dfrac{p^{-3}}{\frac{1}{16}\cdot\frac{1}{x^2}\cdot y^2}$

$\dfrac{4\left(\frac{1}{x^4}\right)y^2p^4}{x} * \dfrac{(1/p^3)}{y^2/16x^2}$

$\dfrac{4y^2p^4}{x^4 \cdot x} * \dfrac{16x^2}{y^2 p^3}$

$(4*16)\left(\dfrac{y^2}{y^2}\right)\left(\dfrac{p^4}{p^3}\right)\left(\dfrac{x^2}{x^4 \cdot x^1}\right)$

$(64)(y^{2-2})(p^{4-3})(x^{2-5})$

$64\,y^0p^1x^{-3}$

$64*1*p\,(1/x^3)$

$\boxed{\dfrac{64p}{x^3}}$

Scientifiction Notation

Scientific notation is a special way of writting very small or large numbers.

A scientific notation uses a combination of numbers between 0 and 9 and powers with base 10.

Exponent.

$= a * 10^6$

a needs to be between 1 and 9

Example:

a) 2,434412
$= 2.434412 * 1,000,000$
$= 2.43412 * 10^6$

b) 240,000
$= 2.4 * 100,000$
$= 2.4 * 10^5$

c) 51
$= 5.1 * 10$
$= 5.1 * 10^1$

For example if we have a very small negative numbers, then, we follow the same principle except the exponent becomes a negative

0.0023

we need to put the decimal between 2 and 3 by moving 3 units to the write

$0.0023 = 2.3 * 10^{-3}$

d) 0.00012

decimal need to be between 1 and 2

→ move 4 unit to the right.

$0.00012 =$
$1.2 * 10^{-4}$

e) 0.00000253

decimal need to be between 2 and 5.

0.00000253
$= 2.53 * 10^{-6}$

Multiplying numbers in scientific notation:

Example:

a) $(2.3 * 10^5) * (1.2 * 10^2)$

$= (2.3 * 1.2)(10^5 * 10^2)$

$= (2.76)(10^{5+2})$

$= 2.76 * 10^7$

b) $(4.2 * 10^{-3}) * (2.5 * 10^3)$

$= (4.2 * 2.5)(10^{-3} * 10^3)$

$= 10.5 * 10^{-3+3}$

$= 10.5 * 10^0$

$= 10.5 * 1$

$= 1.05 * 10^1$

c) $(1 * 10^5)(2.3 * 10^{-2})$

$(1 * 2.3)(10^5 * 10^{-2})$

$2.3\ (10^{5+(-2)})$

$2.3 * 10^3$

$2.3 * 10^3$

d) $(-2 * 10^3) / (4 * 10^{-2})$

$\left(\frac{-2}{4}\right)\left(\frac{10^3}{10^{-2}}\right)$

$(-0.5)(10^{3-(-2)})$

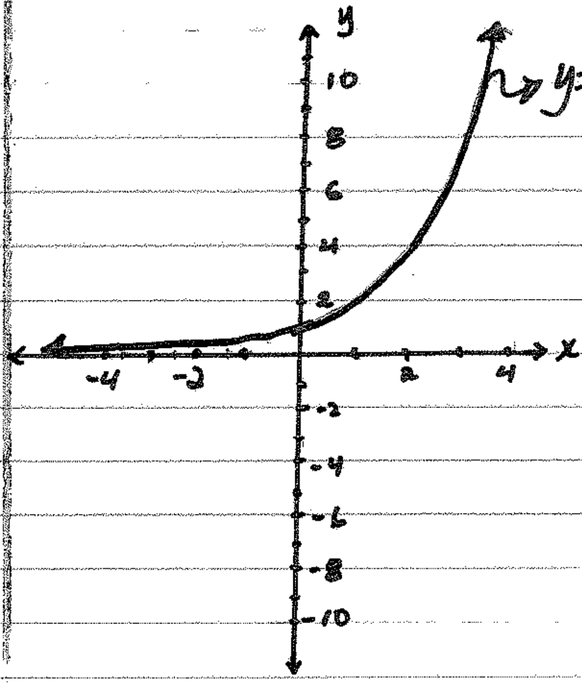
$(-0.5)(10^{3+2})$

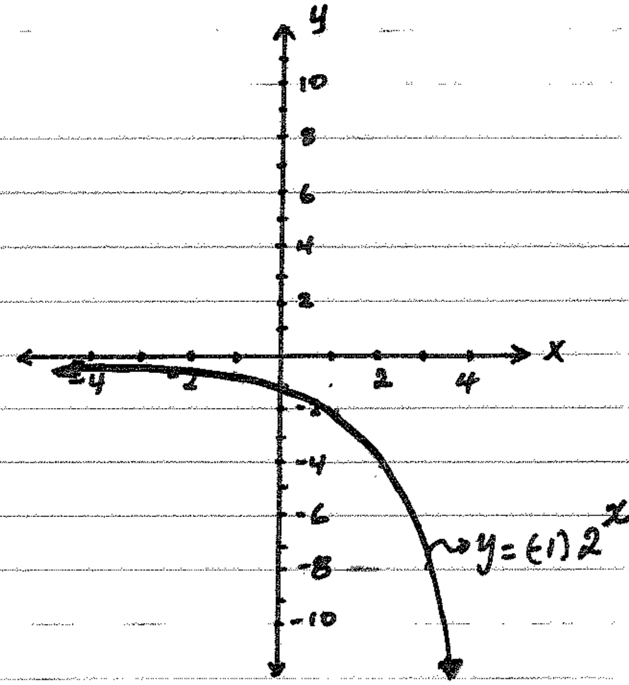
$-0.5 * 10^5$

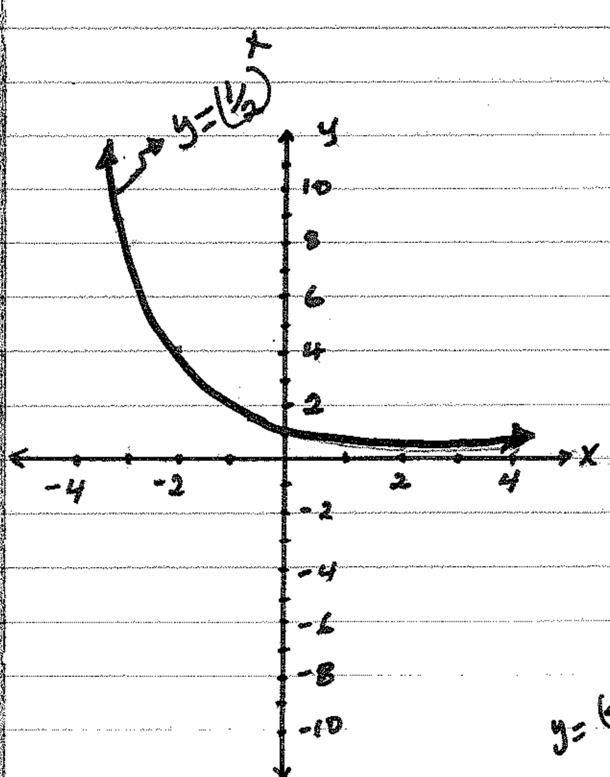
$-5 * 10^4$

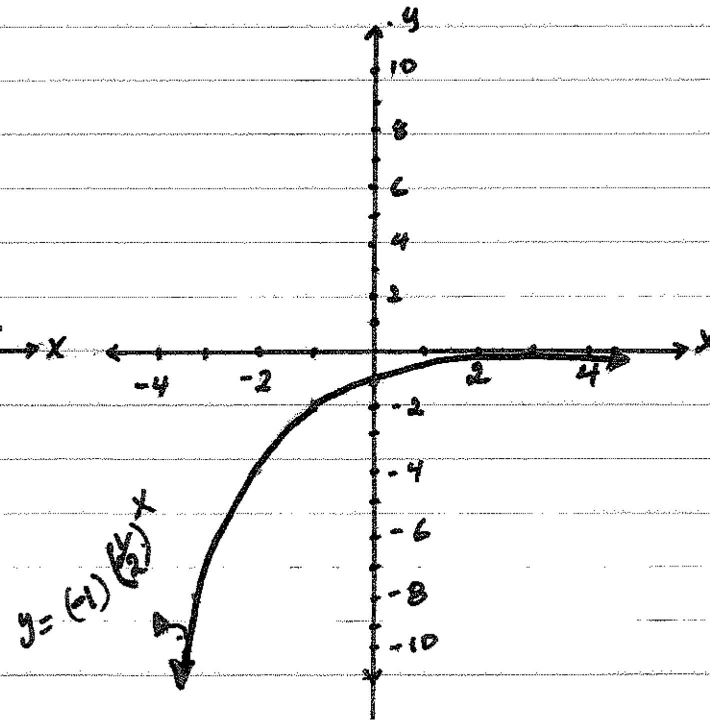
Exponential function and their graphs.

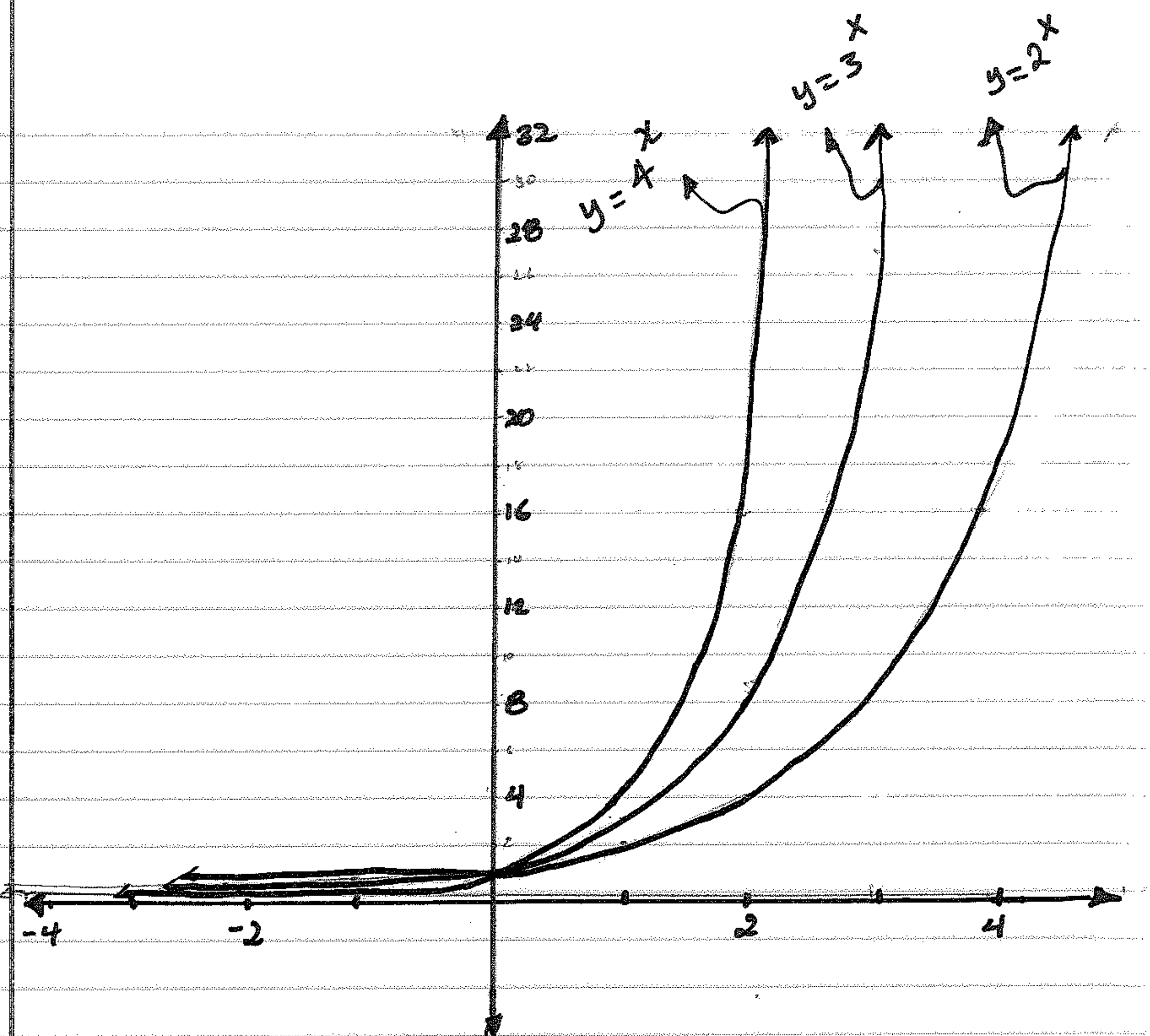
Exponential functions are the form $y = a(b)^x$
where $a \neq 0$ and $b \neq 1$ and $x \in \mathbb{R}$











Notice that the graph gets closer and closer as "b" $y = a(b)^x$ increases from 1 to $\rightarrow \infty$

When b is between 1 and ∞, the exponential function is commonly known as Exponential Growth

When b is between 0 and 1, the exponential function is commonly known as Exponential Decay.

In this case 'b' is called a growth factor and a decay factor respectively.

How to find exponential functions given two points:

Eg 1: Find the function (exponential) that passes through (1, 6) and (3, 24)

Let $f(x) = a(b)^x$ be the function.
Now, we just need to find the values of a and b and we are done.

$y = a * (b)^x$

From the two points, we get two equations

$\overset{x}{(1},\overset{y}{6)}$ $\quad y = a(b)^x$

$6 = a(b)^1$

$6 = ab$

$\overset{x}{(3},\overset{y}{24)}$ $\quad y = a(b)^x$

$24 = a(b)^3$

$24 = ab^3$

Now, we can write the first equation in terms of a, i.e

$\frac{6}{b} = \frac{ab}{b}$

$a = \frac{6}{b}$ --- substitute this in the 2nd Eq.

$24 = ab^3$

$24 = (\frac{6}{b}) b^3$

$24 = 6(\frac{b^3}{b})$

$24 = 6b^{3-1}$

$\frac{24}{6} = \frac{6b^2}{6}$

$b^2 = 4$

$b = \pm\sqrt{4}$

$b = \pm 2$

$a = \frac{6}{b}$

$a = \pm\frac{6}{2}$

$a = \pm 3$

$\Rightarrow$ $\boxed{f(x) = 3(2)^x \text{ OR } f(x) = -3(-2)^x}$

Working with Polynomials

A polynomial function is given by the form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_0$$

It is very important to understand some of the vocabularies and terminologies that are associated with a polynomial function.

$$p(x) = \underbrace{a_n}_{\text{coefficient}} \underbrace{x}_{\text{variable}}{}^{\overbrace{n}^{\text{degree}}} + \underbrace{a_{n-1}\, x^{n-1}}_{\text{term}} + \cdots + \underbrace{a_0}_{\text{constant}}$$

$$\underbrace{a_n, a_{n-1}, a_{n-2}, \cdots, a_0}_{\text{coefficients of the polynomial}}$$

$$\underbrace{n, n-1, n-2, \cdots, 0}_{\text{degrees of the polynomial}}$$

$$\underbrace{a_n x^n, a_{n-1} x^{n-1}, a_{n-2} x^{n-2}}_{\text{terms of the polynomial}}, \cdots$$

a_0

constant, it is also a term.

Examples of polynomials

a) $3x^4 + 2x^5 - 3$
b) $3x^2 - 1$
c) $-\frac{1}{3}x^2 + 2x$
d) $x^5 - 1x^2 + 6$
e) 4

This are some examples of polynomials.

Standard form of a polynomial:
when we write a polynomial, ~~with~~ the term with the highest degree, comes 1st and the rest of the polynomial terms follow in descending order:

For example

$p(x) = 3x^4 + 2x^5 - 3 + 4x$ Not standard form

$\downarrow$

$p(x) = 2x^5 + 3x^4 + 4x - 3$ standard form.

More examples on terminologies.

a) $p(x) = 2x^3 - 4x + 1$

→ has three terms
→ the degree of the polynomial is 3.

b) $p(x) = -4x^4 + 2$

→ has two terms
→ the degree of the polynomial is 4.

c) $p(x) = \frac{1}{2}x + 1$

→ has two terms
→ the degree of the polynomial is 1.

Naming Polynomials

We can name polynomials based on the numbers of terms the polynomial has and the degree of the polynomial:

* Remember the degree of a polynomial is the highest exponent.

Name Based on # of Terms		Name Based on Degree	
1 term	Monomial	0 degree	Constant
2 terms	Binomial	1st degree	Linear
3 terms	Trinomial	2nd degree	Quadratic
4 terms & more	Polynomial	3rd degree	Cubic
		4th degree	Quartic
		5th degree	Quintic
		6th degree & more	6th degree polynomial.

Example:
Name the following polynomials

a) $4x^3 - 2x$ — a Cubic binomial
b) $3x^2 + 2x - 1$ — a Quadratic trinomial
c) x^4 — a Quartic monomial
d) $\frac{1}{2}x^5 + 2x - 1$ — a Quintic trinomial
e) $x^7 + 4$ — a 7th degree binomial
f) -2 — a constant monomial
g) $x^6 - 2x^3 + 4x^2 - 2$ — a 6th degree polynomial

Simplifying Polynomials

Definition:

Like terms:

like terms are terms with the same variables and exponents/degree.

for example,

- $2x^2$ & $-3x^2$
- $4p^3$ & p^3
- $-2q^5$ & $10q^5$
- 7 & 2 etc.

We can always combine like terms based on how they are connected. For example if two like terms are combined by addition, then we can add their respective coefficients

i.e $2x^3+5x^3 \rightarrow 7x^3$

Simplifying polynomials is basically combining like terms, if there are any like terms.

Eg1: $p(x) = 3x^3 \underline{-2x^2} \underline{+4x^2} - x + 2$

there two like terms. $-2x^2$ and $4x^2$

we can combine -2 & 4 and take up one of the x^2's.

$(-2+4)x^2$

$2x^2$

$p(x) = 3x^3 + 2x^2 - x + 2$

Eg2: $f(x) = 2 \underline{+x^2} - x^3 \underline{+2x^2}$

two like terms; x^2 & $2x^2$

$(1+2)x^2$

$3x^2$

$p(x) = \underbrace{2+3x^2-x^3}$ → rearranging the terms to standard form

$f(x) = -x^3+3x^2+2$

Adding and Subtracting Polynomials.

We can add polynomials by adding like terms:

Eg1. Let $f(x) = 2x^3 + x$

$g(x) = -3x^2 + 2x + 1$

$$f(x) + g(x) = (2x^3 + x) + (-3x^2 + 2x + 1)$$

1st lets make sure we get rid of the parenthesis. since there are no numbers multiplying each polynomial, we can just eliminate the parenthesis.

$$= 2x^3 + x + (-3x^2) + 2x + 1$$

$$= 2x^3 + \underline{x} - 3x^2 + \underline{2x} + 1$$

$$= \boxed{2x^3 - 3x^2 + 3x + 1}$$

Eg2: let $g(x) = 10x^4 + 2x^3 - 3$

$u(x) = -10x^4 - 2x^3 + 5$

$$g(x) + u(x) = (10x^4 + 2x^3 - 3) + (-10x^4 - 2x^3 + 5)$$

$$= \underline{10x^4} + \underline{2x^3} - 3 \underline{-10x^4} \underline{-2x^3} + 5$$

like terms

$$= (10x^4 - 10x^4) + (2x^3 - 2x^3) + (-3 + 5)$$

$$= 0 + 0 + 2$$

$$= \boxed{2}.$$

Eg3: Let $f(x) = x^6$

$g(x) = 2x^3 - x + 1$

$$f(x) + g(x) = (x^5) + (2x^3 - x + 1)$$

$$= \boxed{x^5 + 2x^3 - x + 1}$$

Subtracting polynomial

Let $f(x) = x^2+2$

$g(x) = -x^3+2x^2-5$

$f(x)-g(x) = (x^2+2)-(-x^3+2x^2-5)$

↳ notice that, the minus sign needs to be distributed over g(x) before we simplify.

$= (x^2+2)-(-x^3+2x^2-5)$

$= x^2+2-(-x^3)-(2x^2)-(-5)$

$= \underline{x^2}+\underline{2}+x^3-\underline{2x^2}+\underline{5}$

like terms

$= -x^2+7+x^3$

↳ Rearranging standard form.

$= \boxed{x^3-x^2+7}$

Let $f(x) = -x^5+10x^2+3x$

$g(x) = 2x^2+3x$

This time lets subtract f(x) from g(x)

$g(x)-f(x) = (2x^2+3x)-(-x^5+10x^2+3x)$

$= (2x^2+3x)-(-x^5+10x^2+3x)$

$= 2x^2+3x-(-x^5)-(10x^2)-(3x)$

$= \underline{2x^2}+\underline{3x}+x^5-\underline{10x^2}-\underline{3x}$

$= (2x^2-10x^2)+(3x-3x)+x^5$

$= -8x^2+0+x^5$

↳ Rearranging to Standard form

$= \boxed{x^5-8x^2}$

Multiplying Polynomials

Multiplying polynomials is a multiple step process.

Lets start by multiplying a monomial with polynomials:

Eg: $4x^2\ (2x^3+3x^2-1)$

The monomial $(4x^2)$ is going to multiply every term in the trianomial $(2x^3+3x^2-1)$

Distributing $4x^2$

$$4x^2\ (2x^3+3x^2-1)$$

Recall some of the properties of exponents with the same base: $x^3 \cdot x^4 = x^{3+4} = x^7$

$$(4x^2 * 2x^3) + (4x^2 * 3x^2) - (4x^2 * 1)$$
$$(4*2)(x^2 * x^3) + (4*3)(x^2 * x^2) - (4*1)x^2$$
$$8x^{2+3} + 12x^{2+2} - 4x^2$$
$$\boxed{8x^5 + 12x^4 - 4x^2}$$

Eg2: $-3x^4(-2x^2+x^4-1)$

$$-3x^4\ (-2x^2+x^4-1)$$
$$(-3x^4 * -2x^2) + (-3x^4 * x^4) - (-3x^4 * 1)$$
$$(-3*-2)(x^4 * x^2) + (-3*1)(x^4 * x^4) - (-3*1)x^4$$
$$6x^{4+2} + (-3)x^{4+4} - (-3)x^4$$
$$6x^6 - 3x^8 + 3x^4$$

rearranging to standard form

$$\boxed{-3x^8 + 6x^6 + 3x^4}$$

Multiplying Binomials with polynomials

Recall that a binomial has two terms. Each term needs to be distributed over the other polynomial. i.e;

Eg: $(2x^2 - x)(x^3 + 2x^2 - 4)$

$(2x^2 - x)(x^3 + 2x^2 - 4)$

$(2x^2)(x^3 + 2x^2 - 4) \; - x(x^3 + 2x^2 - 4)$

$(2x^2 * x^3) + (2x^2 * 2x^2) - (2x^2 * 4) + (-x * x^3) + (-x * 2x^2) - (-x * 4)$

$2x^{2+3} + 4x^{2+2} - 8x^2 - (x^{1+3}) - (2x^{1+2}) + 4x$

$2x^5 + \underline{4x^4} - 8x^2 \underline{- x^4} - 2x^3 + 4x$

like terms

$$\boxed{2x^5 + 3x^4 - 2x^3 - 8x^2 + 4x}$$

Eg: $(-x^5 + 3x^2)(x^2 - 5x)$

$(-x^5 + 3x^2)(x^2 - 5x)$

$(-x^5 * x^2) - (x^5 * -5x) + (3x^2 * x^2) + (3x^2 * -5x)$

$(-x^{5+2}) - (5x^{5+1}) + (3x^{2+2}) + (-15x^{2+1})$

$-x^7 - 5x^6 + 3x^4 - 15x^3$

Remark:

Always remember the signs of each term when you multiply polynomials.

Factoring Polynomials

Factoring polynomials is the reverse process of multiplying polynomials

Before we go any further with factoring polynomials, let's talk about the greatest common factor (GCF) since it's going to play an important role.

Prime factorization:

factors are numbers that multiply together to give a bigger number.
when factors are prime numbers, then we call the method prime factorization.

Eg: 4 and 3 are factors of 12 but they are not the prime factor of 12. This is because 4 is not a prime number. If we chang 4 into 2 and 2, then 2, 2, & 3 are the prime factors of 12.

$12 = 2 * 2 * 3.$

Eg: $48 = 12 * 4$

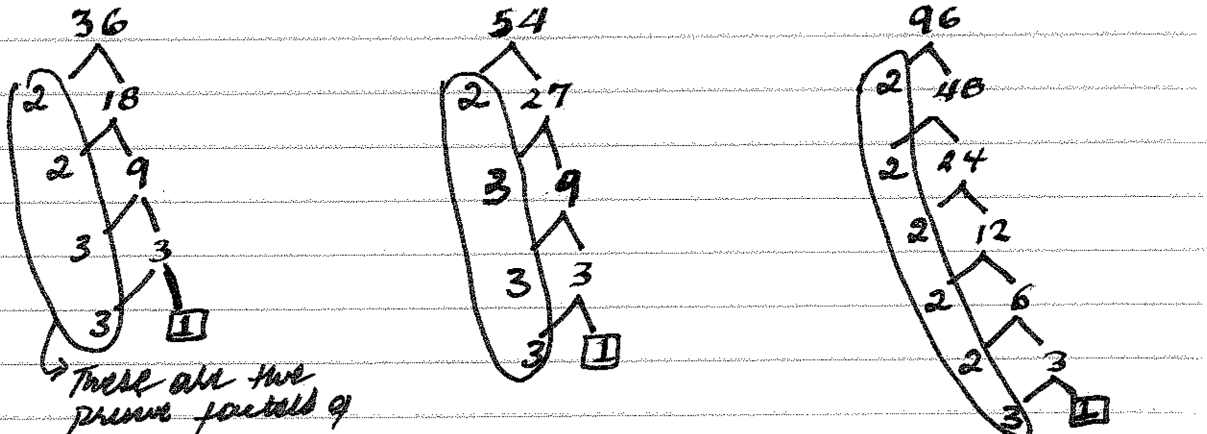
$= \underbrace{2 * 2 * 3} * \underbrace{2 * 2}$

$= 2 * 2 * 2 * 2 * 3$ prime factors of 48.

Factor tree is one of the best ways to find the prime factors of any number.

How: Divid the number starting with 2, 3, .. unit, we end up with 1.

Eg: Factor Tree:



These are the prime factors of

$36 = 2*2*3*3$ $\qquad$ $54 = 2*3*3*3$ $\qquad$ $96 = 2*2*2*2*2*3$

Now that we have ~~seen~~ seen how prime factorization works, let start factoring polynomials.

Eg: $15x^4 + 36x^2$

$15x^4 = \underline{\underline{3}} * 5 * x * x * \underline{x * x}$
$36x^2 = 2 * 2 * \underline{\underline{3}} * 3 * \underline{x * x}$ } let's look for something that both polynomials have

They both have one 3 and two x's. and we can factor them out.

i.e

$15x^4 \quad + \quad 36x^2$

$\underline{\underline{3}} * 5 * x * x * \underline{x * x} + 2 * 2 * \underline{\underline{3}} * 3 * \underline{x * x}$

whenever you factor out GCF, you need to leave the remaining terms inside a parenthesis. i.e

$3 * x * x \; [5 * x * x + 2 * 2 * 3]$

$3x^2 \; [5x^2 + 12]$

If we distribute the $3x^2$ over $5x^2 + 12$, we will get our original terms.

Eg:

$$54x^3 - 18x^2 + 36x$$

$54x^3 = \underline{2 * 3} * 3 * 3 * \underline{x} * x * x$

$18x^2 = \underline{2 * 3} * 3 * x * \underline{x}$

$36x = \underline{2 * 3} * 2 * 3 * \underline{x}$

$GCF = 2 * 3 * x = 6x$

$54x^3 - 18x^2 + 36x = 6x[3*3*x*x - 3*x + 2*3]$

$= 6x[9x^2 - 3x + 6]$

Eg: $15x^{10} - 25x^4 + 45x^2$

$15x^{10} = 3 * \underline{5} * \underline{x * x} * x * x * x * x * x * x * x * x$

$25x^4 = 5 * \underline{5} * \underline{x * x} * x * x$

$45x^2 = 3 \cdot 3 \cdot \underline{5} * \underline{x * x}$

$GCF = 5 * x * x$

$= 5x^2$

Factoring $5x^2$ from $15x^{10} - 25x^4 + 45x^2$ yeilds

$5x^2[3x^8 - 5x^2 + 9]$

If we distribute $5x^2$ over $(3x^8 - 5x^2 + 9)$ we get.

$5x^2(3x^8 - 5x^2 + 9)$

$(5x^2 * 3x^8) - (5x^2 * 5x^2) + (5x^2 * 9)$

$(5*3\,x^{2+8}) - (5*5\,x^{2+2}) + (5*9\,x^2)$

$15x^{10} - 25x^4 + 45x^2$

which is the original polynomial.

That is why, factoring is the reverse process of multiplying polynomials.

Special Cases

1. $(x+y)^2 = x^2+2xy+y^2$
2. $(x-y)^2 = x^2-2xy+y^2$
3. $x^2-y^2 = (x-y)(x+y)$
4. $x^3+y^3 = (x+y)(x^2-xy+y^2)$
5. $x^3-y^3 = (x-y)(x^2+xy+y^2)$

Examples:

Eg 1. $(x+2)^2 = x^2+2(x)(2)+2^2$
$= x^2+4x+4$

Eg 2. $(p-3)^2 =$
$p^2-2(p)(3)+3^2$
p^2-6p+9

Eg 3. $x^2-25 =$
$= x^2-5^2$
$= (x-5)(x+5)$

Eg 4. $x^3+8 =$
$= x^3+2^3$
$= (x+2)(x^2+x(2)+2^2)$
$= (x+2)(x^2+2x+4)$

Eg 5. x^3-64
$= x^3-4^3$
$= (x-4)(x^2+x(4)+4^2)$
$= (x-4)(x^2+4x+16)$

These examples demonstrate how powerful these special case can be, when we need to expand or factor polynomials that satisfies the above conditions.

Factoring Trinomials

$p(x) = ax^2 + bx + c$, with degree 2.
where a, b, and c are integer coefficients

In order to understand factoring polynomials of the form ax^2+bx+c, 1st let's see an example that demonstrates a trinomial of that form and its factors.

$(x+2)$ and $(x+3)$ are factors of x^2+5x+6
if we multiply $(x+2)$ by $(x+3)$, we get

$$(x+2)(x+3)$$

$$\underline{x*x} + \underline{x*3} + \underline{2*x} + \underline{2*3}$$
$$x^2 + \underbrace{3x + 2x} + 6$$
$$x^2 + 5x + 6$$

As a result $(x+2)$ & $(x+3)$ are factors of x^2+5x+6

Now:
The question is how could we find these factors given the trinomial x^2+5x+6

Method called sum & Product.

trinomial form ax^2+bx+c $\qquad ax^2+bx+c$

we look for two numbers when we add them we get "b" and when we multiply them, we get "a*c" $\Rightarrow$

Sum	Product
b	$a*c$

Eg:

$x^2 + 5x + 6$

sum	product
5	$1*6=6$

The numbers that we are looking for are 3 and 2 since

sum	product
$2+3=5$	$2*3=6$

it's convenient if we look for the factors of "6"

$7=1+6$	$1*6=6$
$\boxed{5=2+3 \quad 2*3=6}$	
$-7=-1+-6$	$-1*-6=6$
$-5=-2+-3$	$-2*-3=6$

Now we can replace the $\underline{5x}$ in $x^2+\underline{5x}+6$ by $2x+3x$

$x^2 + 5x + 6$

$x^2 + 2x + 3x + 6$

now we can group x^2+2x and $3x+6$

$\underbrace{x^2+2x} + \underbrace{3x+6}$

- x^2+2x has 'x' as factor

$\underbrace{x(x+2)} + \underbrace{3(x+2)}$ $\quad x^2+2x = x(x+2)$

- $3x+6$ has 3 as a factor

$(x+2)$ is a factor for both $x(x+2)$ and $3(x+2)$

$\underline{x}(x+2) + \underline{\underline{3}}(x+2)$

$(x+2)(x+3)$

summary

	sum	product
x^2+5x+6		
$\underline{x^2+2x+3x+6}$	5	6
$x(\underline{x+2}) + 3(\underline{x+2})$	the #'s 2 & 3	
$(x+2)(x+3)$		

$(x^2+5x+6) = (x+2)(x+3)$

Eg

$x^2 - 4x + 3$ 	**sum** -4 	**product** $1*3 = 3$

#'s -1 and -3

$\underline{x^2 - 1x} \underline{-3x + 3}$

$x(x-1) - 3(x-1)$

$(x-1)(x-3)$

factors: $(x-1)(x-3)$

Eg $x^2 + 8x - 20$ 	**sum** 8 	**product** $1*-20 = -2$

#'s 10 and -2

$8x = -2x + 10x$

$\underline{x^2 - 2x} + \underline{10x - 20}$

$x(x-2) + 10(x-2)$

$(x-2)(x+10)$

Now let's try when the leading coefficient "a" is different from 1.

Eg. $2x^2 + 5x - 7$ 	**sum** 5 	**product** $2*-7 = -14$

#'s 7 and -2

$\underline{2x^2 - 2x} + \underline{7x - 7}$

$2x(x-1) + 7(x-1)$

$(x-1)(2x+7)$

Eg $-3x^2 - 12x + 15$ 	**sum** -12 	**product** $-3*15 = -45$

#'s -15 and 3

$-3x^2 - 15x + 3x + 15$

$-3x(x+5) + 3(x+5)$

$(x+5)(-3x+3)$

Remark: Not every trinomial has factors of these form

Eg. $x^2 + 1x + 2$

doesn't have factors.

Factoring polynomials by Grouping:

Sometimes we can factor polynomial of some form by grouping.

↳ grouping means, gathering terms that have some common factor.

Eg 1:

$3x^4 + 4x^3 + 6x^2 + 8x$

here, we can group $3x^4 + 4x^3$ and $6x^2 + 8x$

$\underbrace{3x^4 + 4x^3} + \underbrace{6x^2 + 8x}$

$\underbrace{x^3(3x+4)}_{=} + \underbrace{2x(3x+4)}_{=}$

$(3x+4)(x^3+2x)$

Eg 2: $\underbrace{2x^5 + 3x^2} \underbrace{- 2x^3 - 3}$

$x^2(2x^3+3) - 1(2x^3+3)$

$(2x^3+3)(x^2-1)$

Eg 3: $x^8 + 4x^4 - 5$

To factor the above polynomial 1st lets substitute $x^4 = q$

$(x^4)^2 + 4(x^4) - 5$

$q^2 + 4q - 5$

↳ now lets use sum & product.

	sum	product
	4	-5
#'s	5, and -1	

$q^2 - 1q + 5q - 5$

$q(q-1) + 5(q-1)$

$(q-1)(q+5)$ → now replace q by x^4

$(x^4-1)(x^4+5)$

Working with Quadratic Equations and functions.

Quadratic functions are a specific types of polynomial functions where the highest degree is 2.

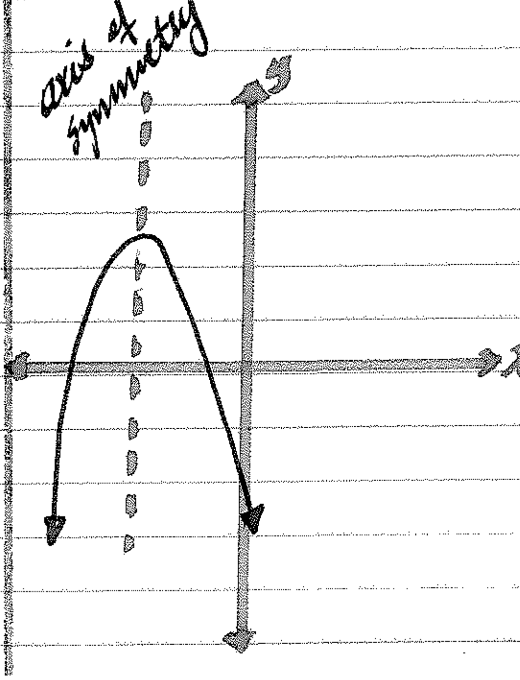
Let $Q(x)$ be a quadratic function. The standard form of a quadratic function is given by:

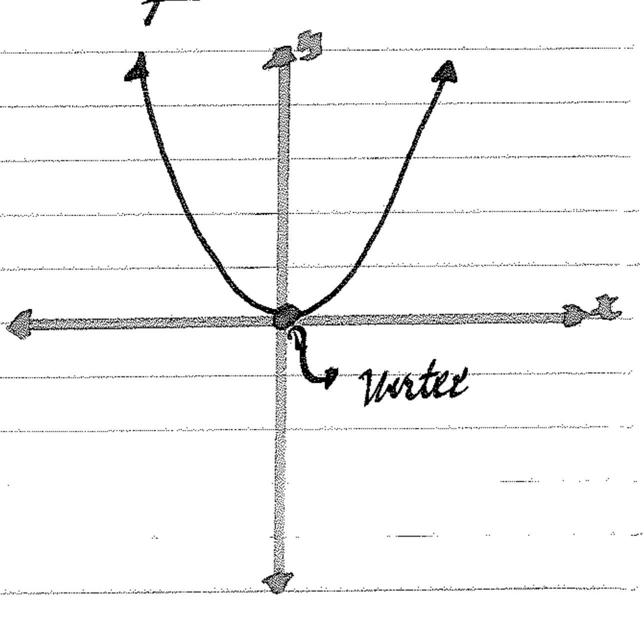
$$Q(x) = ax^2 + bx + c$$

where $a, b, \& c$ are real coefficients. and $a \neq 0$.

notice that, if $a=0$, we don't have an x^2 term, which result "No" quadratic function.

The graph of quadratic functions is called "parabola".





Axis of symmetry: Is a line that divides the parabola in to two equal parts.

Vertex: Is a point a given parabola where the graph turns direction. It is also known as turning point. It is normal given/represented by $V(h, k)$ where "h" is the x coordinate and "k" is the y-coordinate of turning point/vertex.

There are two forms of quadratic functions:

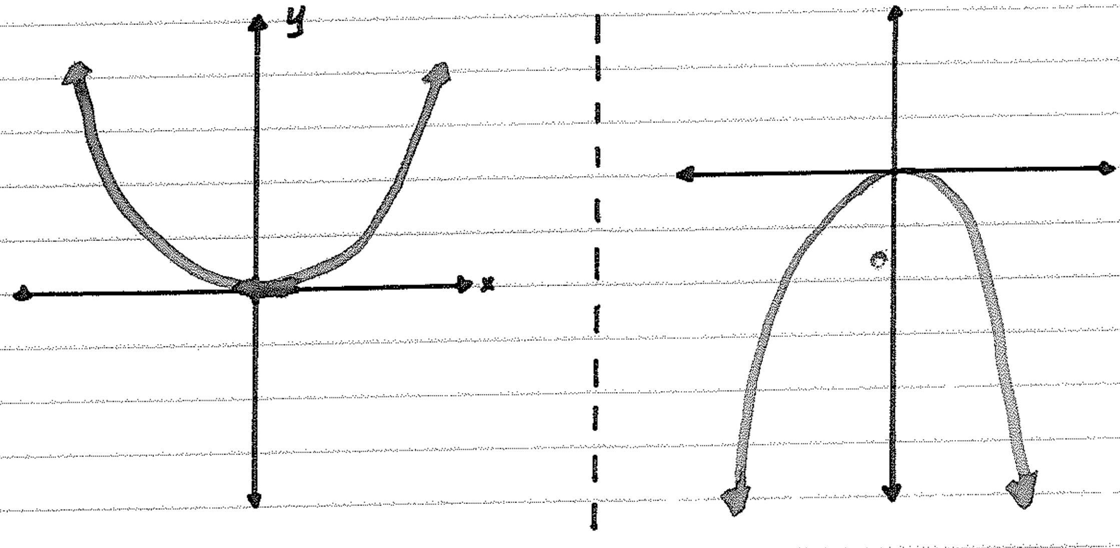
1. standard form: $Q(x) = ax^2 + bx + c$
2. Vertex form: $Q(x) = a(x-h)^2 + k$

Two types of parabola:

upward parabola and downward parabola

if $a > 0$ if $a < 0$

$\Rightarrow$ positive "a" $\Rightarrow$ negative "a"



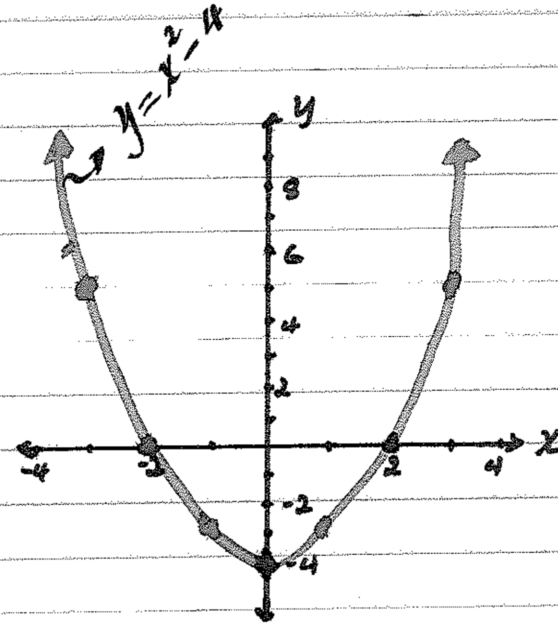
Drawing parabolas:

Eg: Draw the graph of
$Q(x) = x^2 - 4$

we can create a table

x	$Q(x)$
-2	0
-1	-3
0	-4
1	-3
2	0
3	5

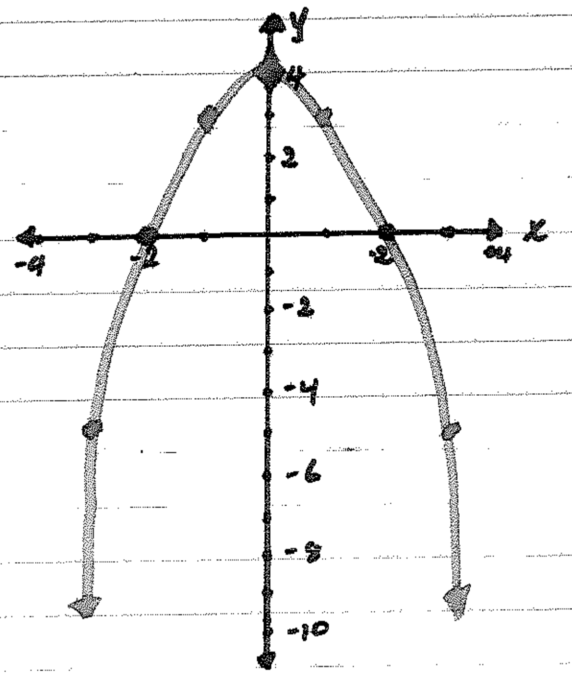
$Q(-2) = (-2)^2 - 4 = 0$
$Q(-1) = (-1)^2 - 4 = -3$
$Q(0) = (0)^2 - 4 = -4$
$Q(1) = (1)^2 - 4 = -3$
$Q(2) = (2)^2 - 4 = 0$
$Q(3) = (3)^2 - 4 = 5$



Eg: Draw the graph of
$Q(x) = -x^2 + 4$

x	$Q(x)$
-3	-5
-2	0
-1	3
0	4
1	3
2	0
3	-5

$Q(-3) = -(-3)^2 + 4 = -5$
$Q(-2) = -(-2)^2 + 4 = 0$
$Q(-1) = -(-1)^2 + 4 = 3$
$Q(0) = -(0)^2 + 4 = 4$
$Q(1) = -(1)^2 + 4 = 3$
$Q(2) = -(2)^2 + 4 = 0$
$Q(3) = -(3)^2 + 4 = -5$



Drawing the graph of a quadratic equation/function is much more ~~easier~~ convinient if the function is given in vertex form. We just need to transform the parent function, which is $Q(x) = x^2$

Axis of Symmetry, vertex and vertex form of Quadratic function.

Recall: Standard form; $Q(x) = ax^2 + bx + c$; $a \neq 0$
Vertex form; $Q(x) = a(x-h)^2 + k$

vertex (V); $V(h, k)$

$\boxed{h = \frac{-b}{2a}}$ $\boxed{k = \frac{4ac - b^2}{4a}}$

Axis of symmetry: $\boxed{x = \frac{-b}{2a}}$

Example:
Find the axis of symmetry and the vertex coordinates for a $Q(x) = x^2 + 8x + 12$.

1st lets write; a, b, and c.

$a = 1$
$b = 8$
$c = 12$

Axis of Symmetry

$x = \frac{-b}{2a} = \frac{-(8)}{2(1)} = \frac{-8}{2} = -4$

AOS; $\boxed{x = -4}$

$h = -\frac{b}{2a} = -\frac{(8)}{2(1)} = \frac{-8}{2} = -4$

$\boxed{h = -4}$

$k = \frac{4ac - b^2}{4a} = \frac{4(1)(12) - (8)^2}{4(1)}$

$= \frac{48 - 64}{4}$

$= \frac{-16}{4}$

$\boxed{k = -4}$

Vertex form:
$Q(x) = a(x-h)^2 + k$
$= 1(x - (-4))^2 + (-4)$
$\boxed{Q(x) = 1(x+4)^2 - 4}$

Eg:

Find the vertex form of the following quadratic function

$Q(x) = 2x^2 + 12x + 10$

1^{st} let's find a, b, and c

$\boxed{a = 2}$

$b = 12$

$c = 10$

$h = \frac{-b}{2a} = \frac{-(12)}{2(2)} = -\frac{12}{4} = -3$

$\boxed{h = -3}$

$k = \frac{4ac - b^2}{4a} = \frac{4(2)(10) - (12)^2}{4(2)} = \frac{80 - 144}{8}$

$= \frac{-64}{8} = -8$

$\boxed{k = -8}$

Vertex form $= a(x-h)^2 + k$

$Q(x) = 2(x-(-3))^2 + (-8)$

$\boxed{Q(x) = 2(x+3)^2 - 8}$

If we expand the vertex form of any quadratic equation / function, we will get its standard form:

Check: $= 2(x+3)^2 - 8$

1^{st} expand $(x+3)^2$

$(x+3)(x+3)$

$x^2 + 3x + 3x + 9$

$x^2 + 6x + 9$

$= 2(x^2 + 6x + 9) - 8$

$= 2x^2 + 12x + \underline{18 - 8}$

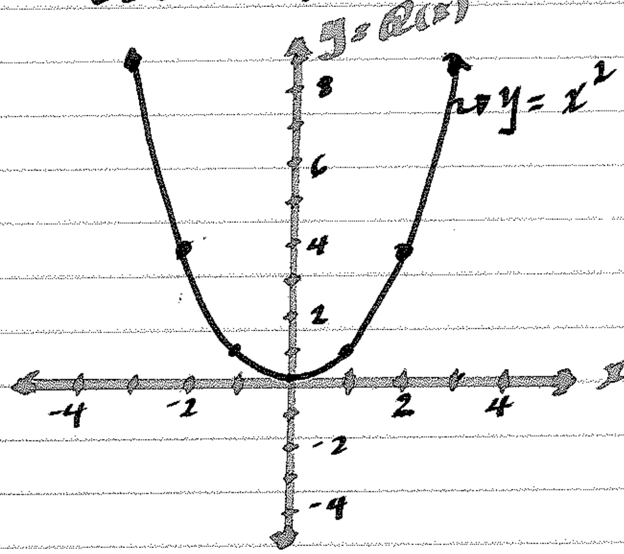
$= 2x^2 + 12x + 10$

which is what we started with.

Transforming a Quadratic function graph.

The parent quadratic function is given by $Q(x) = x^2$

x	$Q(x)$	
-3	9	$= (-3)^2 = 9$
-2	4	$= (-2)^2 = 4$
-1	1	$= (-1)^2 = 1$
0	0	$= (0)^2 = 0$
1	1	$= (1)^2 = 1$
2	4	$= (2)^2 = 4$
3	9	$= (3)^2 = 9$
4	16	$= (4)^2 = 16$



Recall the 1st example that we did.

$Q(x) = x^2 + 8x + 12$

we could draw $Q(x) = x^2 + 8x + 12$ in the same way by creating a table OR we could transform its parent function (i.e $Q(x) = x^2$).

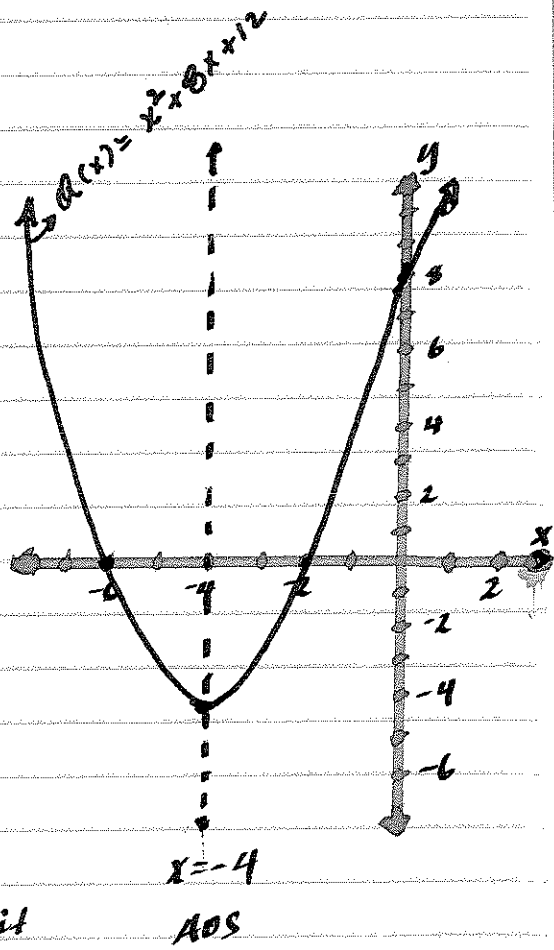
In order to do that, we need its vertex form.

$x^2 + 8x + 12 = 1(x+4)^2 - 4$

In the vertex form

$Q(x) = a(x-h)^2 + k$

shrink or magnify — left or right — up or down



$Q(x) = 1\ (x+4) - 4 \rightarrow$ 4 unit down

The same — 4 unit to the left

Eg:

Draw the graph of
$Q(x) = -2x^2+8x+24$
by translating its parent function i.e ($Q(x)=x^2$)

1st we need to write $Q(x) = -2x^2+8x+24$
in vertex form:

$\boxed{a = -2}$ $\quad h = \frac{-b}{2a} = \frac{-(8)}{2(-2)} = \frac{-8}{-4} = 2$

$b = 8$ $\quad \boxed{h = 2}$

$c = 24$

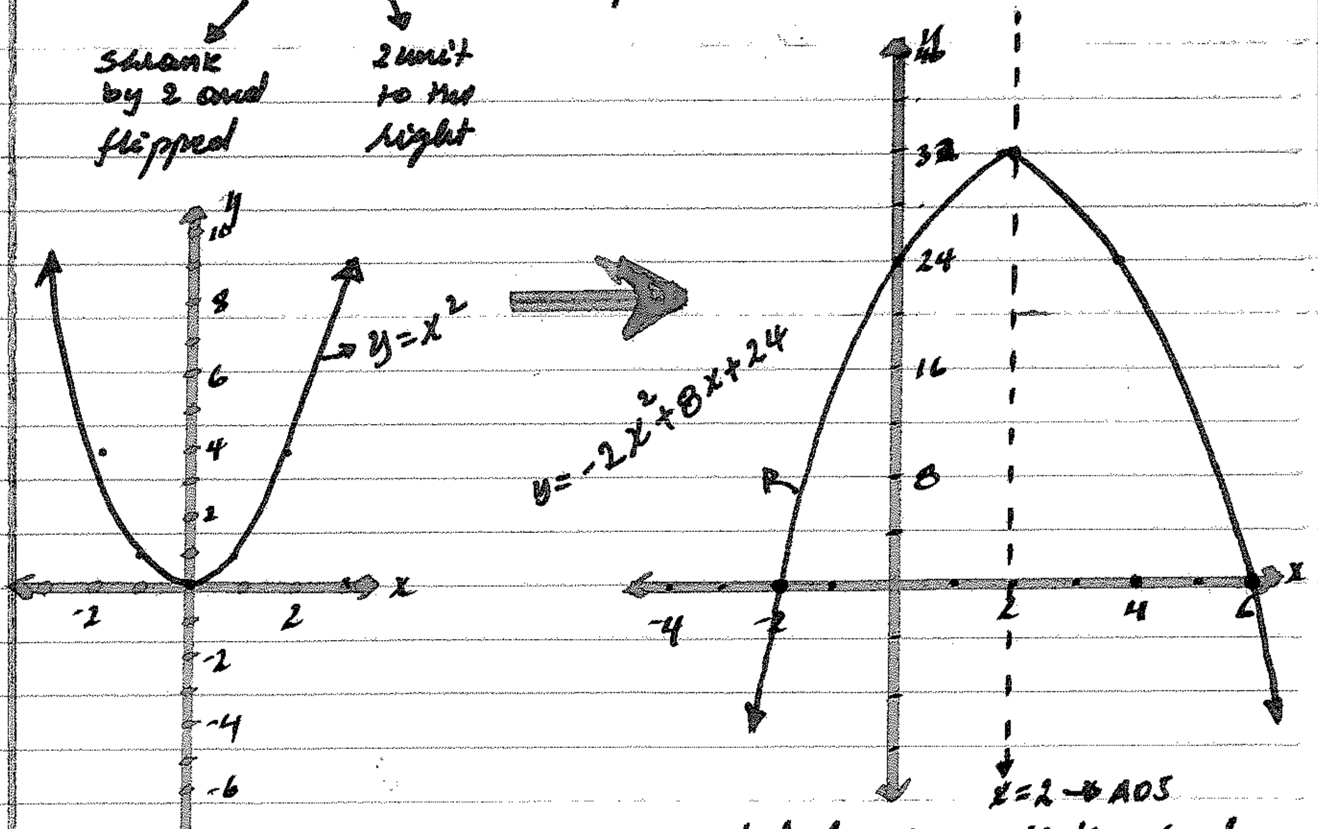
$k = \frac{4ac-b^2}{4a} = \frac{4(-2)(24)-(8)^2}{4(-2)} = \frac{-192-64}{-8} = \frac{-256}{-8}$

$\boxed{k = 32}$

$Q(x) = -2(x-2)^2+32$ shifted 32 units up.

shrank by 2 and flipped

2 unit to the right



Scaled by 4 to fit the graph

Solving Quadratic Equations

Recall: Quadratic functions where given by the form:

$$Q(x) = ax^2+bx+c \quad , a \neq 0$$

$$Q(x) = y$$

$$y = ax^2+bx+c$$

When we set the value of y to 0, then the quadratic functions becomes a quadratic equation.

$$ax^2+bx+c=0$$

↳ we can solve quadratic equation.

Remark:

If it was a quadratic function we can <u>only</u> graph it.

Solving quadratic equation is basically finding value / values of 'x' that makes the equation equals to 0.

Graphically, solutions of quadratic equations are, points where the parabola crosses the x axis, since the value of y equals to zero on the x-axis.

In the previous example

$$Q(x) = -2x^2+8x+24$$

$$\Rightarrow -2x^2+8x+24=0$$

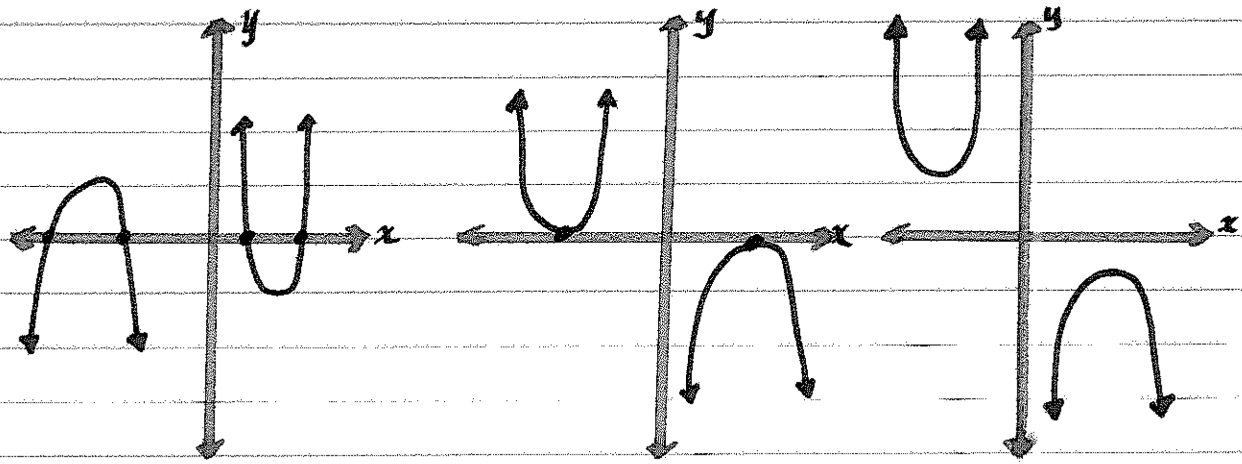
$\boxed{\begin{array}{l} x=-2 \\ x=6 \end{array}}$ and

are the solutions of the quadratic equation $-2x^2+8x+24$.

Type of Solutions

There are three different possible solutions for any given quadratic equation.

Type I: two solutions
Type II: One solution
Type III: no solution



Two sol^ns   One sol^n   No sol^n

Discriminant

The discriminant of a quadratic equation is the value of $b^2 - 4ac$

This will help us to tell which type of solution it has. $D = b^2 - 4ac$

If $D < 0$ (negative) No solution
$D = 0$ One solution
$D > 0$ (positive) Two solutions

The solution set of a quadratic equation is also called "Root".

Techniques solving quadratic equations

1. Factoring
2. Competting the square and
3. Quadratic formula

Factoring:

When we factor a quadratic equation, we get a product of two linear factors.

* When two factors multiply to give zero, one of them must equal to zero, at least. This helps to find the root/roots easily.

Eg 1:

$x^2+5x+6=0$ sum 5   product 6

$x^2+2x+3x+6=0$ #'s 2 and 3

$x(x+2)+3(x+2)=0$

$(x+2)(x+3)=0$

now either $x+2=0$ OR $x+3=0$

$-2 \quad -2$ $-3 \quad -3$

$x=-2$ $x=-3$

$S.S=\{-3,-2\}$

Eg 2: $2x^2-6x+4=0$ sum -6 product $2 \times 4=8$

$2x^2-2x-4x+4=0$ #'s -2 and -4

$2x(x-1)-4(x-1)=0$

$(x-1)(2x-4)=0$

either $(x-1)=0$ OR $2x-4=0$

$+1 \quad +1$ $+4 \quad +4$

$x=1$ $\frac{2x}{2}=\frac{4}{2}$

$x=2$

$S.S=\{1,2\}$

Completing the Square.

Before we go over completing the square, lets talk about a perfect square.

A perfect square is a number or an expression where we can find another whole number or expression multiplied/squared by itself to get the original number.

Eg:
9 is a perfect square because $3*3=9$

15 is not a perfect square since there is no whole number that can be multiplied by itself to give 15.

Eg 49 is a perfect square because $7*7=49$

Eg x^2+4x+4 is a perfect square, because
$(x+2)*(x+2)=x^2+4x+4$.

Now we ready to start completing the square.

Taking the square root of a perfect square.

$$x^2=9$$
$$\sqrt{x^2}=\pm\sqrt{9}$$
$$x=\pm 3$$

to solve for x, we need to get rid of the square by taking the square root of both sides.

$$p^2=64$$
$$\sqrt{p^2}=\pm\sqrt{64}$$
$$p=\pm 8$$

When ever we take square root make sure to put plus/minus "$\pm$" because it could either be positive or negative.

Solve $2x^2+8x-10=0$ by completing the square.

$\frac{2x^2}{2}+\frac{8x}{2}-\frac{10}{2}=\frac{0}{2}$ 1st divide everything by 2.

$x^2+4x-5=0$

$+5 \quad +5$ 2nd add the 5 to the other side

$x^2+4x = 5$

Now we need to add a number to both sides of the equation. The number must make x^2+4x a perfect square such number is found by dividing "b" in this case 4 by 2 and squaring the result

$\frac{4}{2}=2$, $2^2=4$

Add 4 to both sides

$\underbrace{x^2+4x+4}_{\text{This is a perfect square and it can be written as}} = 5+4$

$x^2+4x+4 = (x+2)^2$

$(x+2)^2 = 9$ take the square roots of both sides.

$\sqrt{(x+2)^2} = \pm\sqrt{9}$

$x+2 = \pm 3$

$-2 \quad -2$

$x = -2 \pm 3$

So: $x=-2+3$ and $x=-2-3$

$x=1$ $\quad$ $x=-5$

S.S: $\{-5, 1\}$.

Quadratic formula

We can use completing the square method to derive the quadratic formula.

$$ax^2 + bx + c = 0$$

$$\frac{ax^2}{a} + \frac{bx}{a} + \frac{c}{a} = \frac{0}{a}$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$-\frac{c}{a} \quad -\frac{c}{a}$$

$$x^2 + \frac{b}{a}x + \square = -\frac{c}{a} + \square$$

$$\frac{\frac{b}{a}}{2} = \frac{b}{2a} \Rightarrow \left(\frac{b}{2a}\right)^2$$

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}$$

$$\Downarrow \qquad \times \frac{4a}{4a}$$

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{4ac}{4a^2} + \frac{b^2}{4a^2}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\sqrt{\left(x + \frac{b}{2a}\right)^2} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{\sqrt{4a^2}} \quad ; \quad \sqrt{4a^2} = 2a$$

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$-\frac{b}{2a} \quad -\frac{b}{2a}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

Examples 1:

Solve the quadratic equation $2x^2+2x-24=0$ by using the quadratic formula.

1st lets find the coefficients a, b, c

$a=2$

$b=2$ Formula: $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

$c=-24$

substitute the values of a, b, c $x=\dfrac{-2\pm\sqrt{(2)^2-4(2)(-24)}}{2(2)}$

$=\dfrac{-2\pm\sqrt{4+192}}{4}$

$=\dfrac{-2\pm\sqrt{196}}{4}=\dfrac{-2\pm14}{4}$

S.S = $\{-4, 3\}$ $x=\dfrac{-2-14}{4}$ and $\dfrac{-2+14}{4}$

$x=\dfrac{-16}{4}$ and $\dfrac{12}{4}$

$x=-4$ and $x=3$

Example 2:

Solve by using the quadratic formula. $-3x^2+21x=30$

Before we apply the quadratic we must rewrite the equation in the form of $ax^2+bx+c=0$

Now: $-3x^2+21x=+30$

$-30 \quad -30$

$-3x^2+21x-30=0$

$a=-3$

$b=21$

$c=-30$

$x=2$ and $x=5$

$\boxed{S.S=\{2,5\}}$

$x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

$=\dfrac{-21\pm\sqrt{(21)^2-4(-3)(-30)}}{2(-3)}$

$=\dfrac{-21\pm\sqrt{441-360}}{-6}$

$=\dfrac{-21\pm\sqrt{81}}{-6}$

$=\dfrac{-21\pm9}{-6}$

$x=\dfrac{-21+9}{-6}$ and $x=\dfrac{-21-9}{-6}$

Working with Radical functions and Equations

Before we start taking about radicals, lets review square roots.

square roots are basically, numbers or expressions with rational exponent ($\frac{1}{2}$)

Eg. $\sqrt{4} = (4)^{1/2}$

we can write 4 as 2^2

$(4)^{1/2} = (2^2)^{1/2}$ Recall that we can multiply exponents by exponents $(x^m)^n = x^{mn}$

$= 2^{(2*1/2)}$

$= 2^1$

$= 2$ that is why $\sqrt{4} = 2$.

more example:

$\sqrt{49} = (49)^{1/2}$ $49 = 7*7$

$= 7^2$

$= (7^2)^{1/2}$

$= 7^{(2*1/2)}$

$= 7^1$

$= 7$

⊛ the reason why square roots are $\frac{1}{2}$ rational exponent is because "squares" means "2"

now what if we have a cube root, 4^{th} root, 5^{th} root and etc.

$\sqrt[3]{8}$, $\sqrt[4]{x^8}$, $\sqrt[5]{25}$, ...

$\sqrt[n]{\quad}$ → n^{th} root.

→ look for a number that can be multiplied n times by itself to give the original number.

Radicals

$$\sqrt[n]{\quad}$$

radical symbol also known as Root

n^{th} root also index

Radicand ↳ everything inside the radical symbol.

Simplifying Radicals

Radical expressions are terms that contains radicals. i.e.

$\sqrt{x}, 2\sqrt{5}, \sqrt{3}, -3\sqrt{x+2}, \ldots$

simplify $\sqrt{18}$

If we can find a pair of numbers that are the factors of 18, then we can take one of them out. i.e.

18
2 9
3 3
3 1

$18 = 2*3*3$

$\sqrt{18} = \sqrt{2*\underbrace{3*3}}$ ↳ a pair, so we can take out one of them.

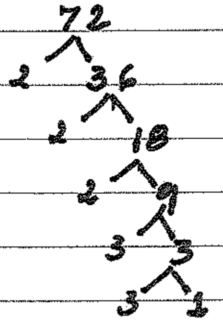
$\boxed{\sqrt{18} = 3\sqrt{2}}$

Remark:

If it was a cube root we would be looking for tripple factors instead of pairs.

Eg.

$\sqrt{72} = ?$



$72 = 2 * 2 * 2 * 3 * 3$

$\sqrt{72} = \sqrt{\underbrace{2*2}*2*\underbrace{3*3}}$

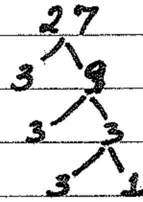
$= 2*3\sqrt{2}$

$= 6\sqrt{2}$

$\boxed{\sqrt{72} = 6\sqrt{2}}$

Eg.

$\sqrt[3]{27}$



$27 = 3 * 3 * 3$

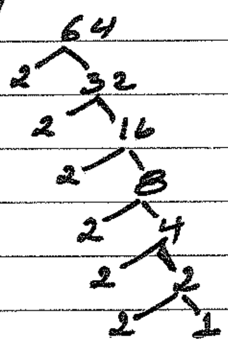
$\sqrt[3]{27} = \sqrt[3]{\underbrace{3*3*3}}$

$= 3$

$\boxed{\sqrt[3]{27} = 3}$

since it is a cube root, we are looking for tripplets.

Eg $\sqrt[5]{64}$



$64 = 2 * 2 * 2 * 2 * 2 * 2$

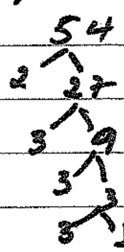
$\sqrt[5]{64} = \sqrt{\underbrace{2*2*2*2*2}*2}$

$= 2\sqrt[5]{2}$

$\boxed{\sqrt[5]{64} = 2\sqrt[5]{2}}$

since it is the 5th root, we are looking for five of a kind

Eg $\sqrt[3]{54x^3y^4}$



$x^3 = x*x*x$

$y^4 = y*y*y*y$

$\sqrt[3]{54x^3y^4} = \sqrt[3]{2*\underbrace{3*3*3}_{3}*\underbrace{x*x*x}_{x}\cdot\underbrace{y\cdot y\cdot y}_{y}\cdot y}$

$\boxed{\sqrt[3]{54x^3y^4} = 3xy\sqrt[3]{2y}}$

Multiplying / Dividing Radicals

Now that we know how to simplify radicals, we can start learning how to multiply and divide them.

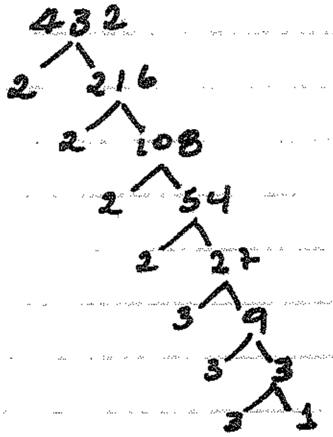
Eg. $\sqrt{24} * \sqrt{18}$

$\sqrt{24*18}$

$\sqrt{432}$

since both terms are square roots, we can combine them under one square roots.

now simplify



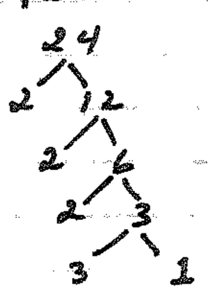
$$\sqrt{432} = \sqrt{\underbrace{2*2}_{2} * \underbrace{2*2}_{2} * \underbrace{3*3}_{3} * 3}$$

$$= 2*2*3\sqrt{3}$$

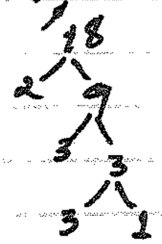
$$\boxed{= 12\sqrt{3}}$$

we also could simplify each radicals before combining them under one radical. i.e

$\sqrt{24}$



$\sqrt{18}$



$$\sqrt{24} = \sqrt{\underline{2*2}*2*3}$$

$$= 2\sqrt{6}$$

$$\sqrt{18} = \sqrt{2*\underline{3*3}}$$

$$= 3\sqrt{2}$$

$$\sqrt{24} * \sqrt{18} = 2\sqrt{6} * 3\sqrt{2}$$

$$= (2*3)\sqrt{6*2}$$

$$= 6\sqrt{3*\underline{2*2}}$$

$$= 6*2\sqrt{3}$$

$$\boxed{= 12\sqrt{3}}$$

Eg:

$\dfrac{\sqrt{24}}{\sqrt{6}}$ we can combine each radical under an radical by division

$$= \sqrt{\frac{24}{6}} = \sqrt{4} = \boxed{2}$$

Eg: $\dfrac{\sqrt{48} * \sqrt{12}}{\sqrt{72}}$

$$= \frac{\sqrt{48*12}}{\sqrt{72}}$$

$$= \frac{\sqrt{576}}{\sqrt{72}}$$

$$= \sqrt{\frac{576}{72}}$$

$$= \sqrt{8} = \sqrt{\underbrace{2*2}*2}$$

$$= 2\sqrt{2}$$

OR: $\sqrt{48} = 4\sqrt{3}$

$\sqrt{12} = 2\sqrt{3}$

$\sqrt{72} = 6\sqrt{2}$

$$\Rightarrow \frac{\sqrt{48} * \sqrt{12}}{\sqrt{72}} = \frac{4\sqrt{3} * 2\sqrt{3}}{6\sqrt{2}}$$

$$= \frac{8\sqrt{3 \cdot 3}}{6\sqrt{2}}$$

$$= \frac{\cancel{8}^{2} * 3^{1}}{\cancel{6}_{2}{}_{1} * \sqrt{2}} = \frac{2}{\sqrt{2}}$$

$$= \underline{\underline{2\sqrt{2}}}$$

Eg: Rationalizing the Denominator.
sometimes some denominators are irrational. The process of changing irrational denominators to rational is called Rationalizing the denominator.

$\dfrac{2}{\sqrt{3}}$ to change the denominator to rational, multiply both the numerator and denominator by $\sqrt{3}$,

i.e

$$\frac{2}{\sqrt{3}} * \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{\sqrt{3 \cdot 3}} = \frac{2\sqrt{3}}{\sqrt{9}} = \boxed{\frac{2\sqrt{3}}{3}}$$

Adding / subtracting Radicals

Unlike multiplying and dividing radicals, adding and subtract can NOT be combined under one radicals.

To add or subtract radicals, 1st we must simplify and see if the terms have the same values under the radical sign.

Eg: $2\sqrt{3} + 4\sqrt{2}$

we can not add this two radicals, since $\sqrt{3} \neq \sqrt{2}$

Eg: $2\sqrt{3} - 4\sqrt{3}$

we can take one of the $\sqrt{3}$ and combine their coefficients accordingly.

$= (2-4)\sqrt{3}$

$= \boxed{-2\sqrt{3}}$

Eg: $5\sqrt{2} + 2\sqrt{2} - 3\sqrt{2}$

$= (5+2-3)\sqrt{2}$

$= \boxed{4\sqrt{2}}$

Eg: $\sqrt{48} + \sqrt{27} - 6\sqrt{3}$

before we decide thes terms don't have a common radicand, let's simplify them

$\sqrt{48} = 4\sqrt{3}$

$\sqrt{27} = 3\sqrt{3}$

$\sqrt{48} + \sqrt{27} - 6\sqrt{3} = 4\sqrt{3} + 3\sqrt{3} - 6\sqrt{3}$

$= (4+3-6)\sqrt{3}$

$= \boxed{1\sqrt{3}}$

Conjugates of Radicals.

Conjugates are the sum and difference of the same radicals.

i.e $2+\sqrt{3}$

its conjugate is $2-\sqrt{3}$

$2\sqrt{3}-\sqrt{2}$

its Conjugate is $2\sqrt{3}+\sqrt{2}$

Multiplying two conjugates will always give us a real number.

i.e. $(2+\sqrt{3})(2-\sqrt{3})$

$$2(2) - 2(\sqrt{3}) + 2(\sqrt{3}) - \underbrace{\sqrt{3*3}}_{3}$$

$$4 - 2\sqrt{3} + 2\sqrt{3} - 3$$

$$4-3$$

$$=1$$

Eg: Rationalize the denominator

$$\frac{4}{\sqrt{5}-\sqrt{2}}$$

to get a rational denominator we need to multiply both the numerator & denominator by the denominator's conjugate, which is $\sqrt{5}+\sqrt{2}$

$$= \frac{4}{\sqrt{5}-\sqrt{2}} \times \frac{\sqrt{5}+\sqrt{2}}{\sqrt{5}+\sqrt{2}}$$

$$= \frac{4(\sqrt{5}+\sqrt{2})}{(\sqrt{5}-\sqrt{2})(\sqrt{5}+\sqrt{2})}$$

$$= \frac{4\sqrt{5}+4\sqrt{2}}{5+\sqrt{10}-\sqrt{10}-2} = \frac{4\sqrt{5}+4\sqrt{2}}{5-2} = \boxed{\frac{4\sqrt{5}+4\sqrt{2}}{3}}$$

Solving Radical Equations.

A Radical equation is an equation involving a variable under the radical sign i.e radicand.

It is very important to talk about "Domain" when we solve radical equations.
This is because, under real numbers, we can not solve negative square roots.
The Domain of a radical equation is all positive numbers including zero.

Find the domain for the following:

a) $\sqrt{x}$; $D = \{x \geq 0\}$

b) $\sqrt{x-2}$;

$x - 2 \geq 0$
$+2 \quad +2$ $\qquad$ $D: \{x \geq 2\}$
$x \geq 2$

c) $\sqrt{5-x}$

$5 - x \geq 0$
$-5 \qquad -5$
$-x \geq -5$
$x \cdot -1 \qquad x \cdot -1$
$x \leq 5$

notice the sign of the inequality switches when we multiply by a negative number.

$D: \{x \leq 5\}$

After we find the domain, solving radicals is getting the variable on one side of the equation by apply the appropriate Algebraic techniques.

Eg: solve:

$\sqrt{x} + 5 = 8$ $\quad$ D: $\{R \geq 0\}$

$\quad -5 \quad -5$

$\sqrt{x} = 3$ $\quad$ square both sides

$(\sqrt{x})^2 = (3)^2$

$x = 9$ $\quad$ the solution is part of the domain.

Eg: $2\sqrt{x} + 2 = 6$

$2\sqrt{x} + 2 = 6$ $\quad$ D: $\{x \in R \geq 0\}$

$\quad -2 \quad -2$

$\frac{2\sqrt{x}}{2} = \frac{4}{2}$

$\sqrt{x} = 2$ $\quad$ square both sides

$(\sqrt{x})^2 = (2)^2$

$x = 4$

Eg: $\sqrt{2x+1} = 5$ $\quad$ Domain: $2x+1 \geq 0$

$(\sqrt{2x+1})^2 = (5)^2$ $\quad -1 \quad -1$

$2x+1 = 25$ $\quad \frac{2x}{2} \geq \frac{-1}{2}$

$\quad -1 \quad -1$

$\frac{2x}{2} = \frac{24}{2}$ $\quad x \geq -1/2$

$x = 12$ $\quad$ D: $\{x \in R \geq -1/2\}$

check:

$\sqrt{2(12)+1} = 5$

$\sqrt{24+1} = 5$

$\sqrt{25} = 5$ ✓

Eg:

$$\sqrt{2x+5} = \sqrt{2-x}$$

$(\sqrt{2x+5})^2 = (\sqrt{2-x})^2$

$2x+5 = 2-x$

$+x \qquad +x$

$3x+5 = 2$

$\quad \cancel{-5} \quad -5$

$\frac{3x}{3} = \frac{-3}{3}$

$\boxed{x = -1}$ the solution is inside the domain.

Domain:

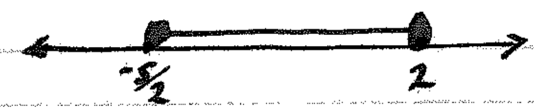
$2x+5 \geq 0$ And $2-x \geq 0$

$-5 \quad -5 \qquad\qquad -2 \quad -2$

$\frac{2x}{2} \geq \frac{-5}{2} \qquad\qquad -x \geq -2$

$\qquad\qquad\qquad\qquad \div -1 \quad \div -1$

$x \geq -5/2 \qquad\qquad x \leq 2$



Eg:

$$\sqrt{5x} + 12 = 2$$

$\sqrt{5x} + 12 = 2$

$\quad -12 \quad -12$

$(\sqrt{5x})^2 = (-10)^2$

$\frac{5x}{5} = \frac{100}{5}$

$x = 20$

Domain:

$\frac{5x}{5} \geq \frac{0}{5}$

$x \geq 0$

Check:

$\sqrt{5(20)} + 12 \stackrel{?}{=} 2$

$\sqrt{100} + 12 \stackrel{?}{=} 2$

$10 + 12 \stackrel{?}{=} 2$

$22 \neq 2$ Not True

"No solution".

Solutions like this one is known as Extraneous solution:

$\Rightarrow$ It is always important to check your answers.

Graphing Radical functions

In order to draw a graph for a radical function, 1st we need to figure out the "Domain" for the function.

This is because, under $\mathbb{R}$, we can "NOT" have negatives under the radical sign.

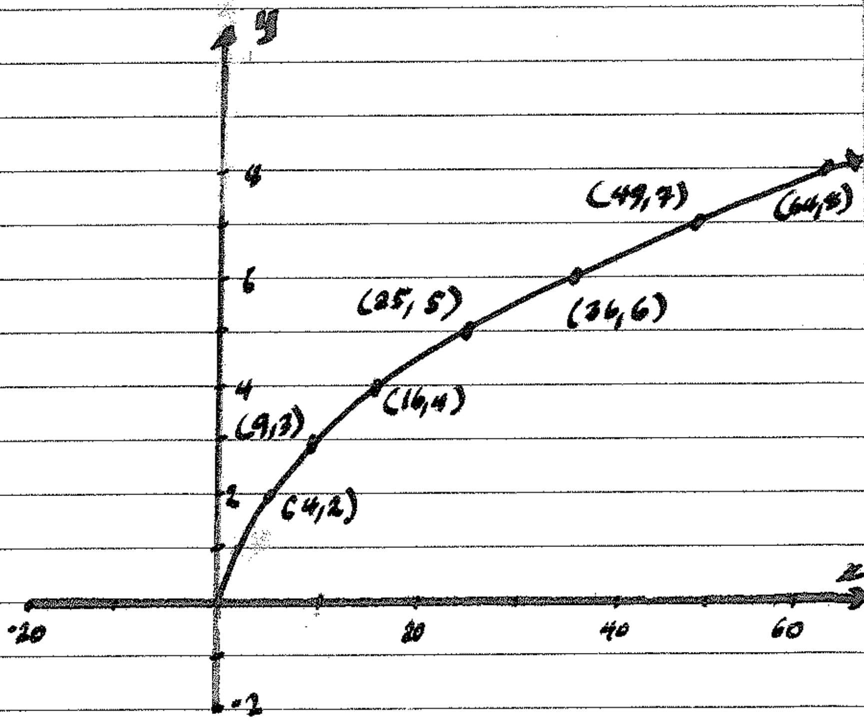
Once we know the domain, we know what kinds of numbers to put for x (input)

Eg: Graph
$$f(x) = \sqrt{x}$$

Domain: $\{x \geq 0\}$

Table

x	$f(x)$	
0	0	$f(0) = \sqrt{0} = 0$
4	2	$f(4) = \sqrt{4} = 2$
9	3	$f(9) = \sqrt{9} = 3$
16	4	$f(16) = \sqrt{16} = 4$
25	5	$f(25) = \sqrt{25} = 5$
36	6	$f(36) = \sqrt{36} = 6$
49	7	$f(49) = \sqrt{49} = 7$
64	8	$f(64) = \sqrt{64} = 8$



In order to fit our table, we also scaled our x-axis.

Notice how I chose my inputs. I only chose numbers that are perfect square.

$f(x) = \sqrt{x}$ is the parent for radical function.

The graph of other radical functions are a transformation of the graph of the parent function

Eg: $a\sqrt{x+b} + c$

a: shrink OR magnify

b: left OR right

c: up OR down

$f(x) = \sqrt{x-2}$ → Its graph is shifted two units to the right

$f(x) = \sqrt{x+5}$ → Its graph is shifted five units to the left.

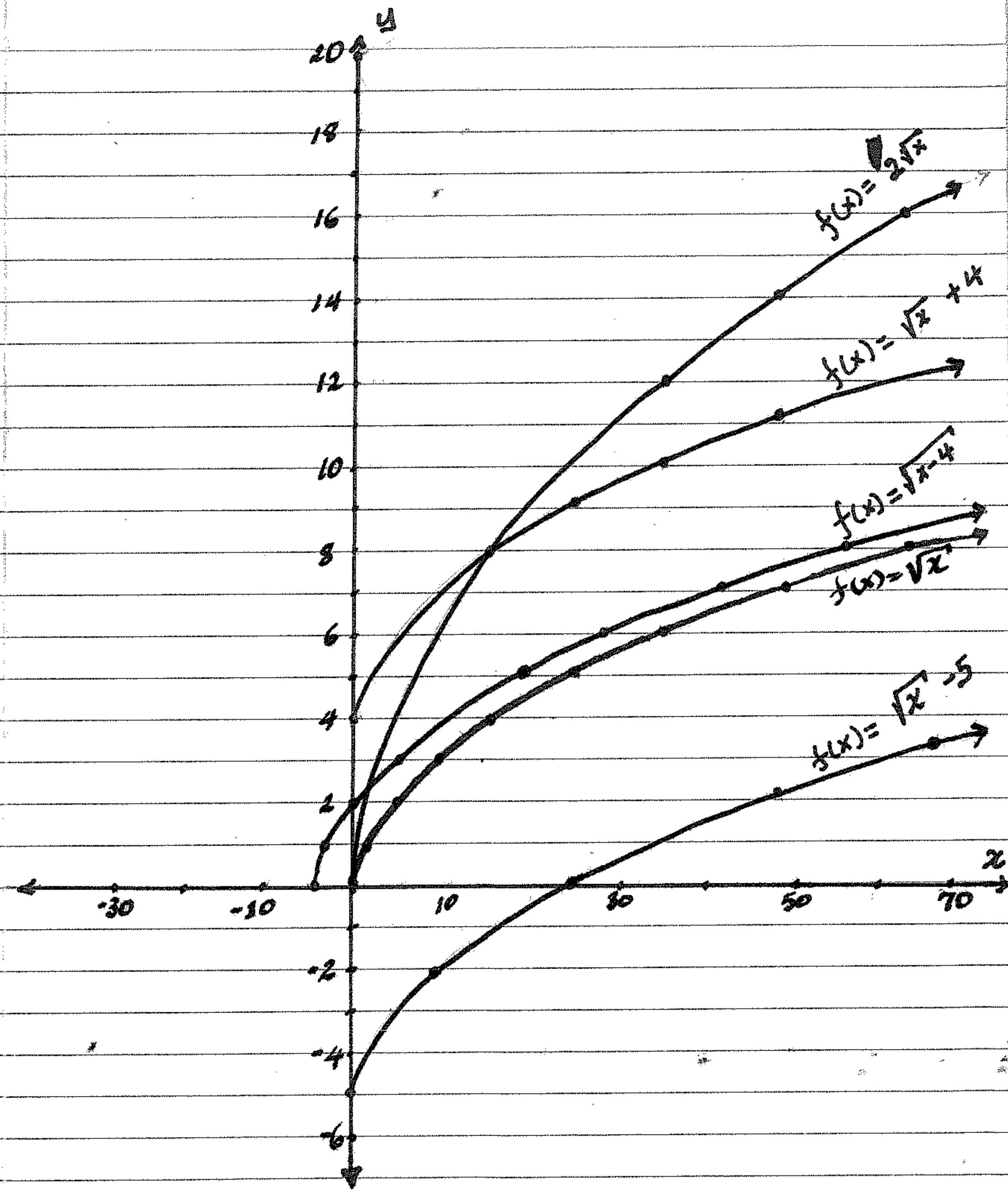
$f(x) = \sqrt{x} + 4$ Its graph is shifted four units up.

$f(x) = \frac{1}{2}\sqrt{x}$ → Its graph is magnified by 2.

$f(x) = 3\sqrt{x}$ → Its graph is shrank by 3.

$f(x) = \frac{1}{4}\sqrt{x+1} - 2$ Its graph is shifted one unit to the left, two units down and magnified by four.

All of these functions' graphs are in comparision with its parent function $f(x) = \sqrt{x}$



The graphs of radical functions demonstrate transformation of the parent function $f(x) = \sqrt{x}$.

Working with Rational functions and Equations.

In this section we are going to study simplifying, solving and graphing rational expressions, equations and functions respectively.

Fractions review:

It is important to go over some fraction concepts. multiplying, dividing, adding, subtracting and simplifying.

Eg: $\frac{12}{18} = ?$ $\quad \frac{\cancel{12}}{\cancel{18}}$ by 2 $\frac{\cancel{6}}{\cancel{9}}$ by 3 $\frac{2}{3}$

$$\frac{12}{18} = \boxed{\frac{2}{3}}$$

Eg: $\frac{14}{15} * \frac{3}{35} = ?$ $\quad \frac{14}{\cancel{15}} * \frac{\cancel{3}}{35}$ by 3 $\frac{\cancel{14}}{5} * \frac{1}{\cancel{35}}$ by 7 $\frac{2}{5} * \frac{1}{5} = \boxed{\frac{2}{25}}$

cross simplifying

Eg: $\frac{24}{16} \div \frac{3}{4}$ we can switch division to multiplication by flipping the divisors

i.e $\quad \frac{24}{\cancel{16}} * \frac{\cancel{4}}{3}$ by 4 $\frac{\cancel{24}}{4} * \frac{1}{\cancel{3}}$ by 3 $\frac{\cancel{8}}{\cancel{4}} * \frac{1}{1}$ by 2 $\frac{2}{1} = \boxed{2}$

Eg: $\frac{4}{3} + \frac{5}{4}$ To add or subtract fractions, we need to have a common denominator

$$\left(\frac{4}{4}\right)\frac{4}{3} + \frac{5}{4}\left(\frac{3}{3}\right) = \frac{4*4}{4*3} + \frac{5*3}{4*3} = \frac{16}{12} + \frac{15}{12}$$

$$= \frac{16+15}{12} = \boxed{\frac{31}{12}}$$

Rational expressions are fractions where the numerators and denominators are polynomials.

Eg: $\frac{2}{x}$, $\frac{3x^2+1}{x^3}$, $\frac{x^2+2}{x^2-4}$, $\frac{x^2+5x+6}{x-2}$ are all rational expressions.

One of the most important things when we work with ration function is making sure that the denominators are different from zero.

If the denominator is zero, then the function becomes undefined, because dividing by zero is not possible.

Figuring out the value of x (variable) that makes the denominator zero helps us to find the domain of the function.

The domain of a rational function is all sets of real number except those values that makes the denominator zero.

Eg: $\frac{1}{x-2}$

To find the domain, set the denominator to equals to zero and find the x value/values that makes it zero.
And exclude those values.

$$x-2=0$$
$$+2 \quad +2$$
$$x=2$$

$$\boxed{\begin{array}{l} D = \{x \in \mathbb{R} \neq 2\} \\ D = \{x \in \mathbb{R} / 2\} \end{array}}$$

Simplifying Rational Expressions.

Just like fraction, if rational expressions have common factor, we can simplify them.

Eg: $\dfrac{2x^3}{x}$.

$$\frac{2x^2 \cdot \cancel{x}}{\cancel{x}} = 2x^3$$

Eg: $\dfrac{3x+6}{x+2}$

$$\frac{3x+6}{x+2} = \frac{3(\cancel{x+2})}{(\cancel{x+2})}$$ factoring 3,

$$= 3$$

Eg: $\dfrac{4x^2}{2x^2+4x}$

$$= \frac{4x^2}{2x^2+4x}$$ we can factor out $2x$ from the denominator

$$= \frac{4x^2}{2x(x+2)}$$ and $4x^2 = 2x * 2x$

$$= \frac{\cancel{2x} \cdot 2x}{\cancel{2x}(x+2)}$$

$$= \frac{2x}{x+2}$$

Eg: $\dfrac{x+2}{x^2-4}$

$x^2-4 = x^2-2^2$

$= (x-2)(x+2)$

$$= \frac{x+2}{x^2-4}$$

$$= \frac{\cancel{(x+2)}}{(\cancel{x+2})(x-2)} = \frac{1}{x-2}$$

Remark:

factoring polynomials play an important role in simplifying rational expressions.

Eg: $\dfrac{2x^3+3x^2+8x+12}{2x^2-x-6}$

Before we try to simplify this expression, lets factor both the numerator and denominator.

	sum.	prod.
$2x^2-x-6$	-1	-12
$2x^2-4x+3x-6$	#s	-4 & 3
$2x(x-2)+3(x-2)$		
$(x-2)(2x+3)$		

$\underbrace{2x^3+3x^2}+\underbrace{8x+12}$

factoring by grouping

$(2x^3+3x^2)+(8x+12)$

$x^2(2x+3)+4(2x+3)$

$(2x+3)(x^2+4)$

$$\frac{2x^3+3x^2+8x+12}{2x^2-x-6} = \frac{\cancel{(2x+3)}(x^2+4)}{(x-2)\cancel{(2x+3)}} = \boxed{\frac{x^2+4}{x-2}}$$

we can not simplify this any further so we stop at $\dfrac{x^2+4}{x-2}$ as the most simplified form.

Multiplying and Dividing Rational expression

Multiplying & dividing rational expressions are similar to multiplying and dividing fractions.

Eg: multiply $\frac{x^2-1}{x}$ by $\frac{x^2}{x-1}$

$$= \frac{(x^2-1)}{x} * \frac{x^2}{x-1}$$

$x^2-1 = x^2-1^2$
$= (x-1)(x+1)$

$x^2 = x \cdot x$

$$= \frac{\cancel{(x-1)}(x+1)}{\cancel{x}} * \frac{\cancel{x}(x)}{\cancel{(x-1)}}$$

$$= \frac{(x+1)}{1} * \frac{x}{1}$$

$$= x^2+x$$

Eg: multiply $\frac{x^2+5x+6}{x^2-4}$ by $\frac{x+2}{x+3}$

$$\frac{(x^2+5x+6)}{(x^2-4)} * \frac{(x+2)}{(x+3)}$$

x^2+5x+6 — sum 5, product 6
$x^2+2x+3x+6$ — #'s are 2 & 3
$x(x+2)+3(x+2)$
$(x+2)(x+3)$

$x^2-4 = x^2-2^2$
$= (x-2)(x+2)$

$$= \frac{\cancel{(x+2)}(x+3)}{\cancel{(x+2)}(x-2)} * \frac{(x+2)}{(x+3)}$$

$$= \frac{\cancel{(x+3)}}{(x-2)} \frac{(x+2)}{\cancel{(x+3)}}$$

$$= \boxed{\frac{(x+2)}{(x-2)}}$$

* Remark: Always make sure that you have simplified the rational expression to their simplified form.

Eg: Divide $\frac{x^3+8}{x-2}$ by $\frac{x+2}{x-2}$

$\frac{x^3+8}{x-2} \div \frac{x+2}{x-2}$, we can convert the division into multiplication by flipping the divisor. i.e

$\Rightarrow \frac{x^3+8}{\cancel{x-2}} * \frac{\cancel{x-2}}{x+2}$

$\Rightarrow \frac{x^3+8}{x+2}$ $\qquad x^3+8 = x^3+2^3 = (x+2)(x^2-2x+4)$

$\Rightarrow \frac{\cancel{(x+2)}(x^2-2x+4)}{\cancel{(x+2)}}$

$\boxed{x^2-2x+4}$

Eg: Divide $\frac{x^2-6x+8}{2x+5}$ by $\frac{2x^2+x-10}{x^2+1}$

$\frac{x^2-6x+8}{2x+5} \div \frac{2x^2+x-10}{x^2+1}$

$\Rightarrow \frac{x^2-6x+8}{(2x+5)} * \frac{(x^2+1)}{2x^2+x-10}$

x^2-6x+8
$x^2-2x-4x+8$
$x(x-2)-4(x-2)$
$(x-2)(x-4)$

$2x^2+x-10$
$2x^2-4x+5x-10$
$2x(x-2)+5(x-2)$
$(x-2)(2x+5)$

$\Rightarrow \frac{\cancel{(x-2)}(x-4)}{(2x+5)} * \frac{(x^2+1)}{\cancel{(x-2)}(2x+5)}$

$\Rightarrow \frac{(x-4)(x^2+1)}{(2x+5)(2x+5)}$

$\Rightarrow \frac{x^3-4x^2+x-4}{(2x+5)^2} = \boxed{\frac{x^3-4x^2+x-4}{4x^2+20x+25}}$

Adding and subtracting Rational expressions:

when we add or subract rational expression, we need to make sure that they have a common denominator.

Technique:

To get the same denominator, check what each denominator is missing to be the same as the other denominator, then multiply both the numerator and denominator by the missing expression.

Eg: $\frac{1}{2x} + \frac{1}{x^2}$

compare $2x$ and x^2
$2x$ needs one more x x^2 needs 2.

$\left(\frac{x}{x}\right)\frac{1}{2x} + \frac{1}{x^2}\left(\frac{2}{2}\right)$

so multiply $\frac{1}{2x}$ by $\frac{x}{x}$ and $\frac{1}{x^2}$ by $\frac{2}{2}$

$\frac{x*1}{x*2x} + \frac{1*2}{x^2*2}$

$\frac{x}{2x^2} + \frac{2}{2x^2}$

now each ration expression has the same dinominator, we can combine / add them

$\frac{x+2}{2x^2}$

since we can nolonger simplify, our answer is $\boxed{\frac{x+2}{2x^2}}$

Eg: add $\frac{x^2+x}{x+1}$ and $\frac{2}{x}$

$\frac{x^2+x}{x+1} + \frac{2}{x}$ multiply $\frac{x^2+x}{(x+1)}$ by $\frac{x}{x}$ and multiply $\frac{2}{x}$ by $\frac{x+1}{x+1}$

$$\left(\frac{x}{x}\right)\frac{x^2+x}{x+1} + \frac{2}{x}\left(\frac{x+1}{x+1}\right)$$

$$\frac{x(x^2+x)}{x(x+1)} + \frac{2(x+1)}{x(x+1)}$$

$$\frac{x^3+x^2}{x^2+x} + \frac{2x+2}{x^2+1}$$

- Now that they have the same denominator, we can add the numerators.

$$\frac{x^3+x^2+2x+2}{x^2+x}$$

$$\frac{\cancel{(x+1)}(x^2+1)}{x\cancel{(x+1)}}$$

$x^3+x^2 + 2x+2$

$x^2(x+1) + 2(x+1)$

$(x+1)(x^2+1)$

x^2+x

$x(x+1)$

$$\boxed{\frac{x^2+1}{x}}$$

In this problem we were able to simplify the rational expressions after adding them. This is because both the numerator and denominator have a common factor i.e $(x+1)$

Subraction:

Eg. subract $\frac{2x}{x+1}$ from $\frac{2x^2}{x^2+4x+3}$

$$\frac{2x^2}{x^2+4x+3} - \frac{2x}{x+1}$$

1st we need to factor x^2+4x+3 in order to see by what expression we need to multiply to get the same denominator.

x^2+4x+3 s   p

$x^2+1x+3x+3$ 4   3

$x(x+1)+3(x+1)$ #'s 3, 1

$(x+1)(x+3)$

$$\frac{2x^2}{x^2+4x+3} - \frac{2x}{x+1}$$

we just need to multiply the 2nd expression by $\frac{x+3}{x+3}$

$$\frac{2x^2}{(x+1)(x+3)} - \frac{2x}{(x+1)}\left(\frac{x+3}{x+3}\right)$$

$$\frac{2x^2}{(x+1)(x+3)} - \frac{2x(x+3)}{(x+1)(x+3)}$$

$$\frac{2x^2}{(x+1)(x+3)} - \frac{2x^2+6x}{(x+1)(x+3)}$$

Now we can subtract the numerators.

$$\frac{2x^2-(2x^2+6x)}{(x+1)(x+3)}$$

$$\frac{\cancel{2x^2}-\cancel{2x^2}-6x}{(x+1)(x+3)} \rightarrow x^2+4x+3$$

$$\boxed{\frac{-6x}{x^2+4x+3}}$$

Solving Rational Equation:

Whenever we solve rational equations, we are looking for values of the variables that will make the equation true.

Sometimes we can not find any solution and sometimes we can find more than solution.

Before we solve rational functions / Equations we need to determine the "Domain"

In this way, if we find solutions that are not in the domain, we can eliminate them.

Eg: $\frac{x-3}{2} = \frac{1}{x}$

Domain $\{ \mathbb{R} \mid x = 0 \}$ x can not be zero.

$\frac{x-3}{2} = \frac{-1}{x}$ now lets make the denominators the same.

$\left(\frac{x}{x}\right)\frac{x-3}{2} = \frac{-1}{x}\left(\frac{2}{2}\right)$

$\frac{x(x-3)}{\cancel{2x}} = \frac{-2}{\cancel{2x}}$ for $x \neq 0$, we can eliminate the denominators

$x^2 - 3x = -2$

$\quad +2 \quad +2$

$x^2 - 3x + 2 = 0$

$x^2 - 2x - 1x + 2 = 0$ $\quad \frac{S}{-3} \quad \frac{P}{2}$

$x(x-2) - 1(x-2) = 0$ $\quad$ #'s $-2\,2\,-1$

$(x-2)(x-1) = 0$

$x - 2 = 0$ or $x - 1 = 0$

$x = 2 \qquad x = 1$

$\boxed{S.S: \{x = 1, 2\}}$

Eg:
Solve $\dfrac{1}{x-4} + \dfrac{1}{x+5} = \dfrac{9}{x^2+x-20}$

1st let's add the left side after finding the domain.

Domain $x-4 \neq 0 \quad x+5 \neq 0$
$x \neq 4 \quad x \neq -5$

$$\left(\frac{x+5}{x+5}\right)\frac{1}{x-4} + \frac{1}{x+5}\left(\frac{x-4}{x-4}\right) = \frac{9}{x^2+x-20}$$

$$\frac{x+5}{(x+5)(x-4)} + \frac{x-4}{(x+5)(x-4)} = \frac{9}{x^2+x-20}$$

$\downarrow$

x^2+x-20 $\quad \dfrac{S}{1} \quad \dfrac{P}{-20}$
$x^2-4x+5x-20$
$x(x-4)+5(x-4)$ #'s 5 & -4

$$\frac{x+5}{(x+5)(x-4)} + \frac{x-4}{(x+5)(x-4)} = \frac{9}{(x+5)(x-4)} \quad (x-4)(x+5)$$

$$\frac{x+5+x-4}{(x+5)(x-4)} = \frac{9}{(x+5)(x-4)}$$

$$\frac{2x+1}{\cancel{(x+5)(x-4)}} = \frac{9}{\cancel{(x+5)(x-4)}}$$ for $x \neq -5$ & 4, we can cancel the denominators.

$2x+1 = 9$
$\quad -1 \quad -1$

$\dfrac{2x}{2} = \dfrac{8}{2}$

$x = 4 \rightarrow$ But $x=4$ is not in our domain. As a result there is No solution.

Extraneous Solution.

Eg:

solve $\dfrac{\dfrac{2x^2+x-1}{x^2+x-6}}{\dfrac{1}{x-2}+\dfrac{1}{x+3}} = 0$

Domain:
we have 3 individual fractions.
1. $x^2+x-6 \neq 0$
2. $x-2 \neq 0$
3. $x+3 \neq 0$

$\Rightarrow D: \{R \mid x \neq -3, 2\}$

lets factor $2x^2+x-1$ and x^2+x-6

$2x^2+x-1$

S	P
-1	-2

$\#\ -2, 1$

$2x^2-2x+1x-1$

$2x(x-1)+1(x-1)$

$(x-1)(2x+1)$

x^2+x-6

S	P
1	-6

$\#\ 3, -2$

$x^2+3x-2x-6$

$x(x+3)-2(x+3)$

$(x+3)(x-2)$

$$\frac{\dfrac{2x^2+x-1}{x^2+x-6}}{\dfrac{1}{x-2}+\dfrac{1}{x+3}} = 0 \qquad \frac{\dfrac{(x-1)(2x+1)}{(x+3)(x-2)}}{\left(\dfrac{x+3}{x+3}\right)\dfrac{1}{x-2}+\dfrac{1}{x+3}\left(\dfrac{x-2}{x-2}\right)} = 0$$

$$= \frac{\dfrac{(x-1)(2x+1)}{(x+3)(x-2)}}{\dfrac{x+3}{(x+3)(x-2)}+\dfrac{x-2}{(x+3)(x-2)}} = 0$$

$$= \frac{\dfrac{(x-2)(2x+1)}{\cancel{(x+3)(x-2)}}}{\dfrac{x+3+x-2}{(x+3)\cancel{(x-2)}}} = 0 \quad \text{for } x \neq -3 \ \&\ 2$$

$$= \frac{(x-2)\cancel{(2x+1)}}{\cancel{(2x+1)}} = 0 \quad \text{for } x \neq -1/2$$

Extraneous solution

$x-2=0$

$\boxed{x=2}$

Ans $\boxed{\text{No solution}}$

Graphs of Rational functions:

Rational functions are one of these functions, where their domain have some restrictions.

This implies that their graphs are <u>not continuous</u> through out the real numbers

These restritions are values of x's which makes the denominator zero/undefined. and horizontal lines which the graph can not cross.

Def: lines where ~~the~~ a rational function graph gets closer and closer but never cross or touch are called $\boxed{\text{Asymptotes.}}$

There are two types of asymptotes.

1. vertical asymptote $x = a,\ a \in \mathbb{R}$
2. Horizontal asymptote $y = k,\ k \in \mathbb{R}$

↳ sometimes oblique $y = mx + b$

(V.A.)
Vertical Asymptotes:
Any values of domain restritions are vertical asymptotes

(H.A.)
Horizontal asymptotes.
1. $y = 0$, if the degree of the denominator is bigger than the degree of the numerator.

2. If the degree of the numerator and denominator is the same, then $y=$ the ratio of the coefficients of the term with the highest degree.

3. If the degree of the numerator is bigger than the denominator, then $y=$ the quotient of the numerator and denominator.

$\boxed{\text{Eg:}}$

$$f(x) = \frac{2x}{-x^3+1}$$

V. A.

$$-x^3+1=0$$
$$-1 \quad -1$$
$$\frac{-x^3}{x-1} = \frac{-1}{x-1}$$
$$\sqrt[3]{x^3} = \sqrt[3]{1}$$
$$x = 1$$
$$\boxed{x=1}$$

H. A.

since the degree of the denominator is bigger than the degree of the numerator

$$\boxed{y=0}$$

$$f(x) = \frac{3x-1}{2x+3}$$

$$2x+3=0$$
$$-3 \quad -3$$
$$\frac{2x}{2} = \frac{-3}{2}$$
$$x = -3/2$$
$$\boxed{x=-3/2}$$

The degrees are the same, so lets find the ratio of their coefficients

$$\boxed{y=3/2}$$

$$f(x) = \frac{x^2+5}{x}$$

$$\boxed{x=0}$$

This is an oblique asymptote:

$$\begin{array}{r|l} & x \\ \hline x & x^2+5 \\ & -x^2 \\ \hline & 5 \end{array}$$

$$\boxed{y=x}$$

Ex:

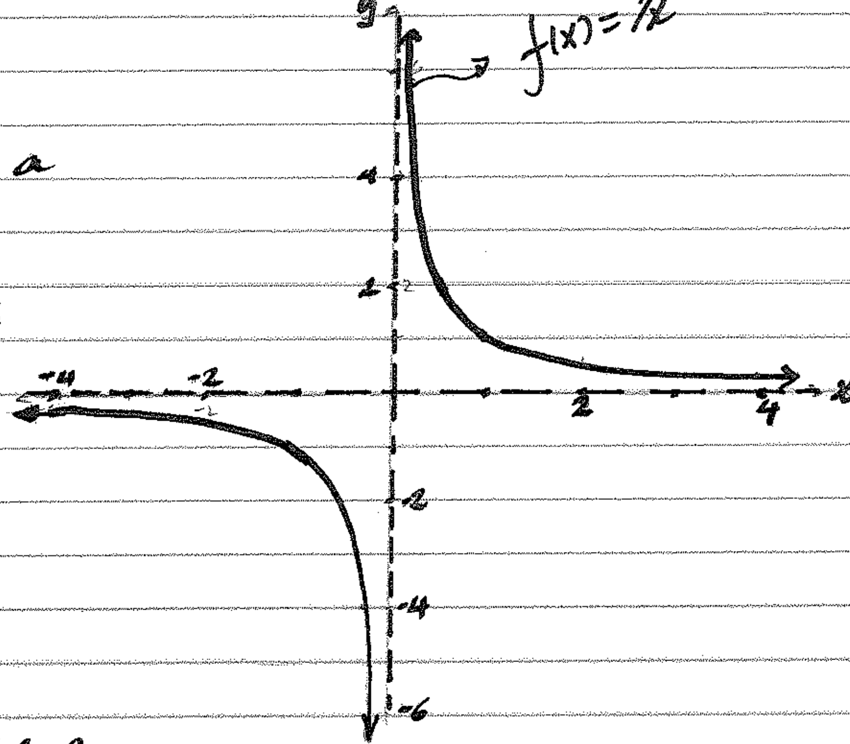
Graph $f(x) = \dfrac{1}{x}$

V.A. $x=0$

H.A. $y=0$

we could create a table.

x	$f(x)$	
-4	$-1/4$	$f(-4) = -1/4$
-2	$-1/2$	$f(-2) = -1/2$
-1	-1	$f(-1) = -1$
0	$\nexists$	$f(0) = \nexists$
1	1	$f(1) = 1$
2	$1/2$	$f(2) = 1/2$
4	$1/4$	$f(4) = 1/4$



1st draw the V.A. & Horizontal A. by using dashed line (i.e ----)

Notice that the graph gets closer and closer to the lines $x=0$ & $y=0$ but never touch or cross those lines.

Observation:

As x approaches to '0' from the left, the graph gets very smaller and smaller and end up approaching $-\infty$.

As x approaches to '0' from the right, the graph gets very larger and larger and end up approaching $+\infty$.

Example:
Draw the graph of $f(x) = \dfrac{-2x^2}{x^2-4}$

V.A: $x^2-4=0$

$x^2-2^2=0$

$(x-2)(x+2)=0$

V.A: $\boxed{x=-2,\ x=2}$

H.A: the degrees are the same

$\Rightarrow$ H.A: $y=-2/1$

$\boxed{y=-2}$