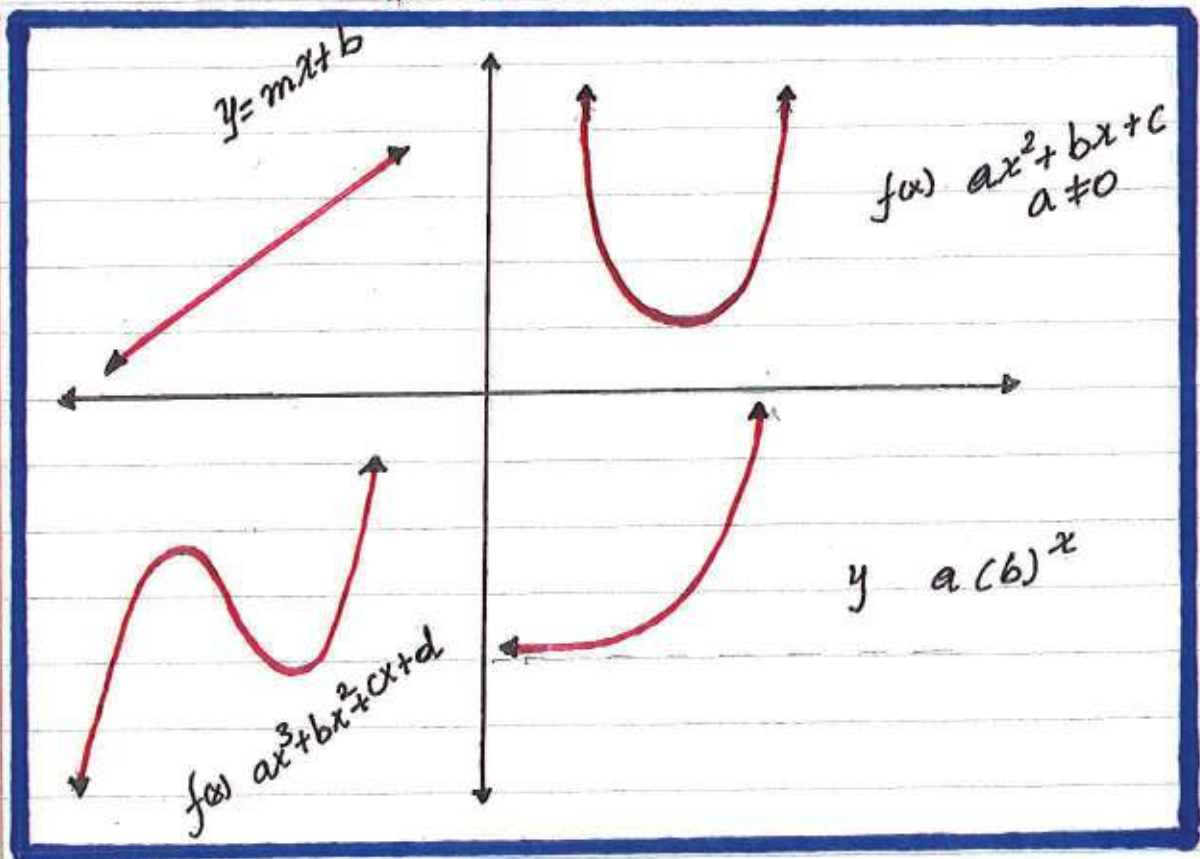


ALGEBRA



FIRST EDITION
BY: TESHALE BYAN

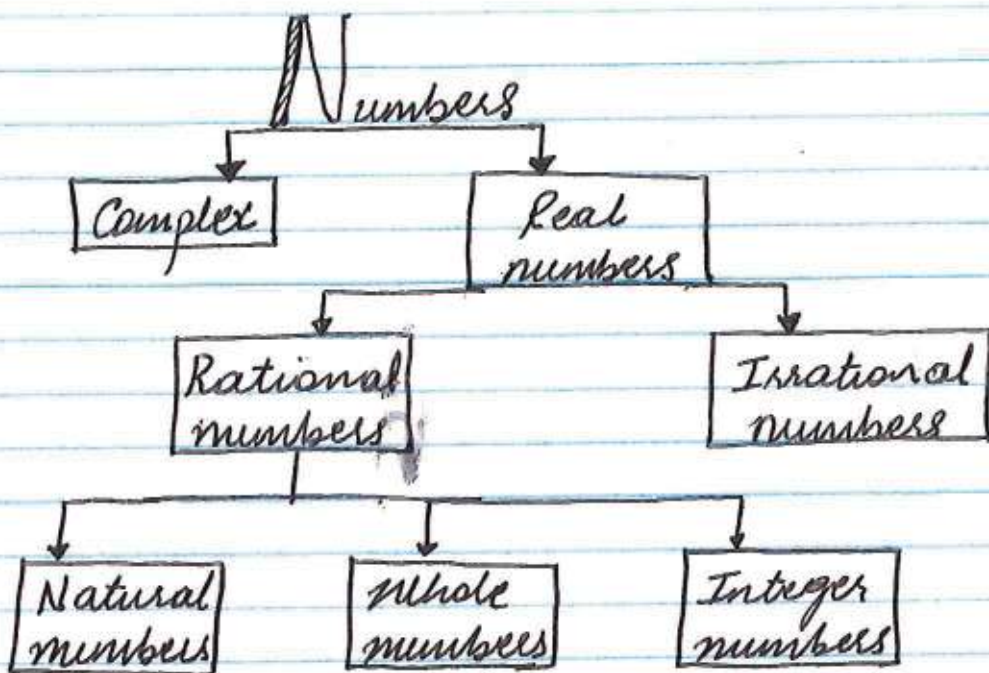
Why math

At some point in every child's life, there is a time where math made sense and hence fun. It might be adding numbers or drawing shapes. It might be solving problems or modeling situation, in any case it was at least interesting. So, where did it become less interesting or even challenging?

Most children share the same experience in terms of challenging math. Once variables are introduced to the math world, things start to get challenging and hence difficult to understand.

Naturally, we tend to understand concrete objects, where we can actually touch and feel. A good example would be counting objects. The next thing will be visualizing objects. A good example would be coloring objects, i.e. categorizing object with similar color/properties. The third piece of this puzzle would be the abstract part. Normally, variables are used to represent abstract part of math.





Even numbers:

Integers that can be divided by 2
with out a remainder

Odd numbers:

Integers that are not divisible by 2.

Prime numbers:

Are natural numbers greater than 1 that has
only two divisors i.e 1 and itself.

2 is the only even prime number.

Composite numbers:

Are natural numbers that have at least
one divisor other than 1 and themselves.

Operations.

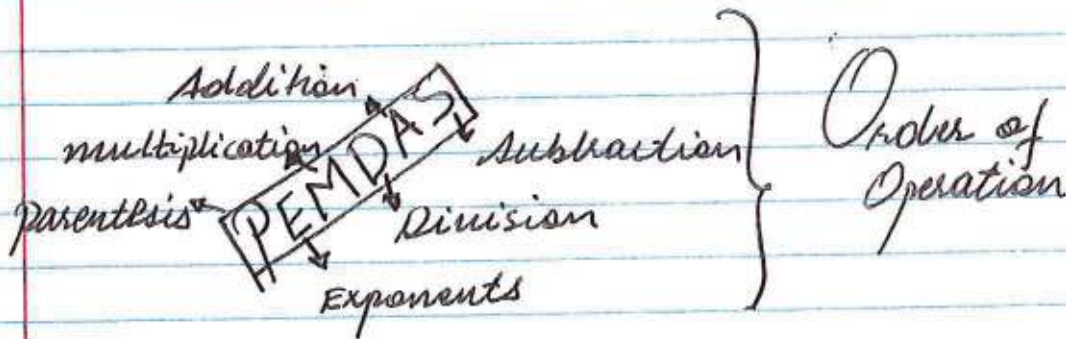
Addition
Subtraction
multiplication
Division

} Basic mathematical operations

Parenthesis
exponents

} other / common mathematical operations

Symbols (+) addition
(-) subtraction
(*) multiplication
(÷) or (/) division
() parenthesis



Algebraic Expression

Numbers
Coefficients
Variables

Example

$$2x^2 + 1$$

Algebraic Equation

Two algebraic expressions connected by the equal sign.

Example

$$2x^2 + 1 = x + 1$$

Evaluate is for an algebraic expression
Solve is for an algebraic equation.

Examples

① Evaluate the expression

$$2x^2 + 1 \text{ for } x=2 \\ x=-5$$

Solution

$$\begin{aligned} 2x^2 + 1 \\ 2(2)^2 + 1 \\ 2(4) + 1 \\ 8 + 1 \\ \underline{\underline{9}} \end{aligned}$$

$$\begin{aligned} 2x^2 + 1 \\ 2(-5)^2 + 1 \\ 2(25) + 1 \\ 50 + 1 \\ \underline{\underline{51}} \end{aligned}$$

② Evaluate the expression

$$-3x + 2(x + 2y) - 5 \text{ for } \begin{array}{l} \text{a) } x=1 \text{ \& } y=-2 \\ \text{b) } x=3 \text{ \& } y=5 \end{array}$$

$$\begin{aligned} \text{a) } & -3x + 2(x + 2y) - 5 \\ & -3(1) + 2(1 + 2(-2)) - 5 \\ & -3 + 2(1 - 4) - 5 \\ & -3 + 2(-3) - 5 \\ & -3 + (-6) - 5 \\ & -3 - 6 - 5 \\ & \underline{\underline{-14}} \end{aligned}$$

$$\begin{aligned} \text{b) } & -3x + 2(x + 2y) - 5 \\ & -3(3) + 2(3 + 2(5)) - 5 \\ & -9 + 2(3 + 10) - 5 \\ & -9 + 2(13) - 5 \\ & -9 + 26 - 5 \\ & -9 + 21 \\ & \underline{\underline{12}} \end{aligned}$$

③ Solve the equation

$$\begin{aligned} 2x + 5 &= -21 \\ -5 & \quad -5 \\ \hline 2x &= -26 \\ \frac{2x}{2} &= \frac{-26}{2} \\ \underline{\underline{x}} &= \underline{\underline{-13}} \end{aligned}$$

$$\begin{aligned} -2x - 3(x + 2) &= 19 \\ -2x - 3x - 6 &= 19 \\ -5x - 6 &= 19 \\ +6 & \quad +6 \\ \hline -5x &= 25 \\ \frac{-5x}{-5} &= \frac{25}{-5} \\ \underline{\underline{x}} &= \underline{\underline{-5}} \end{aligned}$$

Fractions

Fractions are any number that can be written as $\frac{a}{b}$; $a, b \in \mathbb{Z}$ & $b \neq 0$
 \hookrightarrow integers

Examples: $\frac{2}{3}, \frac{1}{2}, \frac{-3}{5}, \frac{5}{1}, \dots$

Arithmetic & Fractions

Multiplying two fractions

notation $\frac{a}{b}$ \rightarrow numerator
 \rightarrow denominator

When we multiply two fractions, we multiply the numerators of the 1st fraction with the numerators of the 2nd fraction and we do the same thing with their denominators:

$$\text{Eg 1. } \frac{3}{4} \times \frac{5}{6} = \frac{3 \times 5}{4 \times 6} = \frac{15}{24}$$

Now we can simplify

$$\Rightarrow \frac{15}{24} = \frac{5}{8} //$$

$$\text{Eg 2. } -\frac{1}{2} \times \frac{4}{3}$$

$$= \frac{-1 \times 4}{2 \times 3} = \frac{-4}{6} \text{ simplify } \frac{-4}{6} = \frac{-2}{3} //$$

$$\text{Eg 3. } \frac{2}{5} \times \frac{4}{6} \times \frac{-1}{2}$$

$$= \frac{2 \times 4 \times -1}{5 \times 6 \times 2} = \frac{-8}{60} \text{ simplify } \frac{-8}{60} = \frac{-2}{15} //$$

Dividing two fractions:

Dividing two fractions is one more step compared with multiplying two fractions.

Key word: Reciprocal

the reciprocal of any fraction is swapping the numerator by the denominator.

Eq: $\frac{2}{5} \rightarrow$ reciprocal will be $\frac{5}{2}$

* Dividing by a fraction is the same as multiplying by the reciprocal of the divisor:

Eq 1. $\frac{5}{3} \div \frac{10}{6}$ \swarrow swap
 $= \frac{5}{3} \times \frac{6}{10} = \frac{5 \times 6}{3 \times 10} = \frac{30}{30} = \frac{1}{1} = 1$

Eq 2. $\frac{\frac{4}{5}}{\frac{2}{15}} \Rightarrow \frac{4}{5} \div \frac{2}{15}$
 $= \frac{4}{5} \times \frac{15}{2} = \frac{4 \times 15}{5 \times 2} = \frac{60}{10} = \frac{6}{1} = 6$

Eq 3. $\frac{1}{3} \div \frac{2}{5}$
 $\Rightarrow \frac{1}{3} \times \frac{5}{2} = \frac{1 \times 5}{3 \times 2}$
 $= \frac{5}{6}$

Subtracting/Adding fractions

In order to understand adding/subtracting fractions, 1st let's try to compare two fractions.

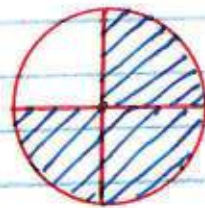
$$\frac{2}{3} \text{ and } \frac{3}{4}$$

These two fractions don't have a common denominator, the 1st fraction ($\frac{2}{3}$) tells us 2 out of 3 and the 2nd fraction ($\frac{3}{4}$) tells us 3 out of 4.



$$\frac{2}{3}$$

Unless both fractions are split among equal parts, it's very difficult to compare.



$$\frac{3}{4}$$

Eg.

Assume you have 50 U.S. dollars and your friend has 100 Chinese yen. The question is, who has more money in England, if you both were there at the same time.

We can not answer/tell for sure whom has more unless we convert your money into pounds.

$$1 \text{ pound} \cong 150 \text{ yen}$$

$$1 \text{ pound} \cong 2 \text{ USD}$$

⇒ you have about 25 pounds
your friend has about 1 pound

Now you can easily tell that you have more money.

⚡ The same thing works when we compare fractions. We need to have the SAME DENOMINATORS.

$\frac{2}{3}$ the denominator is 3
 $\frac{3}{4}$ the denominator is 4

there are a few different techniques that can be used to create a common denominator.

1st: Finding the LCM (Least Common Multiple)
 This is how it's done:

multiples: (x) 1 2 3 4 5 - - - -
 1st denominator: 3: 3 6 9 **12** 15
 2nd denominator: 4: 4 8 **12** 16 20

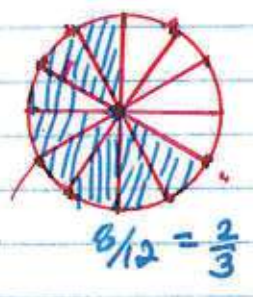
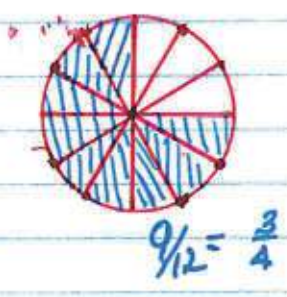
the LCM of 3 & 4 is 12.
 $LCM(3, 4) = 12$

1st fraction $\frac{2}{3}$ need to be multiplied by $\frac{4}{4}$ and
 2nd fraction $\frac{3}{4}$ need to be multiplied by $\frac{3}{3}$.

Remember that $\frac{4}{4} + \frac{3}{3}$ equals 1, so we are not affecting the integrity of both fractions, we are converting them into equivalent fractions:

$\frac{2}{3} \times \frac{4}{4} = \frac{8}{12}$ $\frac{3}{4} \times \frac{3}{3} = \frac{9}{12}$

Now we can compare $\frac{8}{12}$ and $\frac{9}{12}$



Eg 1.

$$\frac{2}{3} + \frac{3}{4}$$

, notice that they don't have a common denominator

⇒ multiply both fractions by the opposite denominators i.e

$$\frac{2}{3} * \frac{4}{4} + \frac{3}{4} * \frac{3}{3}$$

$\frac{2}{3}$ by $\frac{4}{4}$ and $\frac{3}{4}$ by $\frac{3}{3}$

$$\frac{8}{12} + \frac{9}{12}$$

now they share a common denominator so, take one of the denominators and combine the numerators i.e

$$\frac{8+9}{12}$$

$$= \frac{17}{12}$$

Since 17 is a prime, we can NOT further simplify the result.

Eg 2.

$$\frac{8}{6} - \frac{3}{5}$$

1st find their common denominator, i.e multiply $\frac{8}{6}$ by $\frac{5}{5}$ & $\frac{3}{5}$ by $\frac{6}{6}$.

$$\frac{8}{6} * \frac{5}{5} - \frac{3}{5} * \frac{6}{6}$$

$$= \frac{40}{30} - \frac{18}{30}$$

$$= \frac{40-18}{30}$$

$$= \frac{22}{30}$$

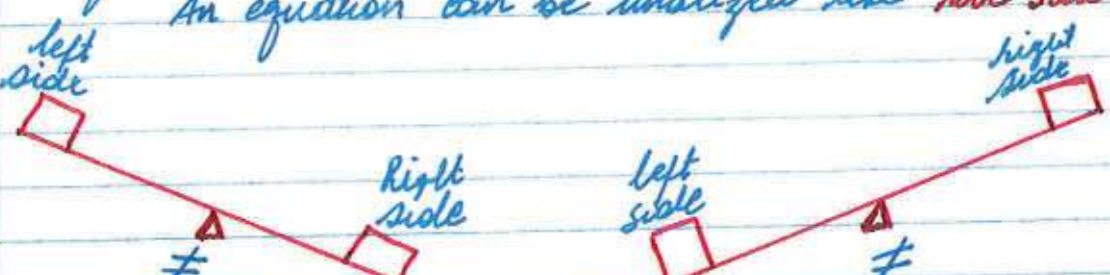
Now simplify the fraction

$$= \frac{11}{15}$$

Solving Equations

Any equation has two sides separated by an equal sign. left side of an equation and right side of an equation.

An equation can be analyzed like *see saw swing*



left side right side

\neq



When an equation is given, both the left & right sides of the equation are balanced. It is very important to preserve these properties.

what ever change we make to either sides of the equation, must be done to the other side of the equation.

An equation normal has the following:

- Variables
- operations
- Numbers
- Equal sign

when we solve an equation (single variable) our ultimate goal is to find a value/values of a variable that makes/make the equation true.

One way to do this is, to leave the variable on one side of the equation alone and take everything to the other side by following the proper mathematical steps & rules.

When we evaluate/simplify an expression, we follow the rule of "PEMDAS". Now, when we solve an equation, we can follow the exact opposite route. "SABMEP".

Example:

Eg1.

$$\begin{array}{ccc} x+4 & = & 10 \\ \text{left} & & \text{right} \\ \text{side} & & \text{side} \end{array}$$

we want to leave x by itself on the left side of the equation.
 \Rightarrow subtracting 4 from both sides

$$\begin{array}{ccc} x+4 & = & 10 \\ -4 & -4 & \end{array}$$

$$\begin{array}{c} x = 6 \\ \hline \end{array}$$

\rightarrow This is also called one step equation solving.

Eg2.

$$2x+4 = 10$$

\rightarrow we want to leave x by itself on the left side. notice that 4 is connected with x by addition & 2 is connected with x by multiplication.

$$\begin{array}{ccc} 2x+4 & = & 10 \\ -4 & -4 & \end{array}$$

$$\begin{array}{ccc} 2x & = & 6 \\ \frac{2}{2} & & \frac{6}{2} \end{array}$$

$$\begin{array}{c} x = 3 \\ \hline \end{array}$$

\rightarrow Two steps equation solving

Eg3. $\frac{x}{2} + 4 = 10$

$$\begin{array}{ccc} \frac{x}{2} + 4 & = & 10 \\ -4 & -4 & \end{array}$$

$$2 \times \frac{x}{2} = 6 \times 2$$

$$\begin{array}{c} x = 12 \\ \hline \end{array}$$

Multi-step & Equations with variables on both sides of the equal sign:

Before we try to solve a value of a variable that makes a given equation true, we need to make sure that all the variables present in the equation are combined together with the proper mathematical steps:

Combining like terms:

$$\begin{array}{r} 2x + 3x \\ 5x \end{array} \quad \begin{array}{r} -3x + x \\ 2x \end{array} \quad \begin{array}{r} 1 + 4 \\ 5 \end{array} \quad \begin{array}{r} 2x - 1 + x \\ 3x - 1 \end{array}$$

Eg1.

$$2x + 1 = 5x - 5$$

$$2x + 1 = 5x - 5$$

$$\begin{array}{r} -1 \\ -1 \end{array}$$

$$2x = 5x - 6$$

$$\begin{array}{r} -5x \\ -5x \end{array}$$

$$-3x = -6$$

$$\begin{array}{r} -3x = -6 \\ -3 \quad -3 \end{array}$$

$$\begin{array}{r} -3x = -6 \\ -3 \quad -3 \end{array}$$

$$\underline{x = 2}$$

1st we need to combine the like terms i.e. $2x$ & $5x$, 1 & -5

now we can divide both sides by "-3" to get the value of "x".

Eg2: $2(-3x+1)-2 = 3x-4(2+x)$

1st distribute the 2 on the left side and -4 on the right side over the parenthesis.

$$2(-3x+1)-2 = 3x-4(2+x)$$

$$-6x+2-2 = 3x-8-4x$$

$$-6x = -4x-8$$

$$\begin{array}{r} +1x \\ +1x \end{array}$$

$$\begin{array}{r} -5x = -8 \\ -5 \quad -5 \end{array}$$

$$x = 8/5 = 1\frac{3}{5}$$

• Combine like terms
• add $1x$ to both sides

When we have relatively complex equations, we need to simplify each side of the sides before we combine like terms.

Simplifying always follow PEMDAS
Solving Equation is the reverse process of simplification.

Eg3. $\frac{1}{2}x + 2(x-1) = \frac{2}{5}x - 3$

distribute 2 over $(x-1)$

$$\frac{1}{2}x + 2x - 2 = \frac{2}{5}x - 3$$

$$\frac{1}{2}x + \frac{4}{2}x - 2 = \frac{2}{5}x - 3$$

• Now we can combine $\frac{1}{2}x + 2x$,
Remember $2x = \frac{4}{2}x$

$$\frac{1+4}{2}x - 2 = \frac{2}{5}x - 3$$

$$\frac{5}{2}x - 2 = \frac{2}{5}x - 3$$

• add 2 to both sides

$$\frac{5}{2}x = \frac{2}{5}x - 1$$

• subtract $\frac{2}{5}x$ from both sides

$$\frac{5}{2}x - \frac{2}{5}x = -1$$

• to subtract $\frac{2}{5}$ from $\frac{5}{2}$ we need to have the same denominator.

$$\left(\frac{5}{2}\right)\frac{5}{2}x - \left(\frac{2}{5}\right)\frac{2}{5}x = -1$$

$$\frac{25}{10}x - \frac{4}{10}x = -1$$

$$\frac{25-4}{10}x = -1$$

$$\frac{21}{10}x = -1$$

• multiply both sides by $\frac{10}{21}$ to get x by itself!

$$\left(\frac{10}{21}\right)\frac{21}{10}x = -1\left(\frac{10}{21}\right)$$

$$x = -\frac{10}{21}$$

Inequalities:

Equations, normally have a specific value/values for their variable that make them true.

Inequalities don't have a specific value, rather they have a set of possible values, that make the whole inequality true.

Inequality
signs

$>$ "greater than"

\geq "greater than or equal to"

$<$ "less than"

\leq "less than or equal to"

After solving Inequalities, it's customary to show the solution on a number line.

Eg.

let $x > -2$
is a solution.



$x \geq -2$



Solid circles

\leq \geq

open circles

$<$, $>$

* Open circle means, the solution does NOT include the start point but everything above/below it.

* Closed circle means, the solution INCLUDES the starting point and the rest.

When we multiply or divide inequalities by a negative number, the inequality sign must also be multiplied/divided by negative, which reverses/switch the sign.

$$\begin{aligned} \text{i.e.} \quad & -> = < & -< = > \\ & -> = \leq & -\leq = \geq \end{aligned}$$

Solving inequalities:

Eq 1. $x + 4 > 2$

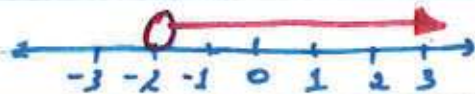
$$\begin{array}{r} x + 4 > 2 \\ -2 \quad -2 \end{array}$$

subtract 2 from both sides

$$\begin{array}{r} x + 2 > 0 \\ -2 \quad -2 \end{array}$$

add -2 to both sides

$$\begin{array}{r} x > -2 \\ \hline \end{array}$$



Eq 2. $3 - 2x \leq -7$

$$\begin{array}{r} 3 - 2x \leq -7 \\ -3 \quad -3 \end{array}$$

subtract 3 from both sides

$$-2x \leq -10$$

divide both side by -2

$$\begin{array}{r} -2x \leq -10 \\ \hline -2 \quad -2 \end{array}$$

* remember dividing by -2

switch the inequality sign.

$$\begin{array}{r} x \geq 5 \\ \hline \end{array}$$



Eq 3. $2x + 3 \leq x$

$$\begin{array}{r} 2x + 3 \leq x \\ -2x \quad -2x \end{array}$$

subtract 2x from both sides

$$\begin{array}{r} 3 \leq -x \\ -1 \quad -1 \end{array}$$

divide both sides by -1, switch the sign.

$$\begin{array}{r} -3 \geq x \\ x \leq -3 \end{array}$$



Multi-step Inequality

Eg1.

$$2(3x-5)+4 \leq -3x+3(x-1)$$

• 1st distribute the 2 & 3 over the parenthesis

$$2(3x-5)+4 \leq -3x+3(x-1)$$

$$6x-10+4 \leq -3x+3x-3$$

• Combine like terms on each side

$$6x-6 \leq -3$$

$$6x-6 \leq -3 \quad \cdot \text{add 6 to both sides}$$

$$+6 \quad +6$$

$$6x \leq 3 \quad \cdot \text{divide by 6 both sides}$$

$$\frac{6x}{6} \leq \frac{3}{6}$$

$$x \leq \frac{1}{2}$$



Eg2.

$$-2(x-1)+3x-1 \geq -2(4x+1)+9x$$

$$-2(x-1)+3x-1 \geq -2(4x+1)+9x$$

$$-2x+2+3x-1 \geq -8x-2+9x$$

$$-2x+3x+2-1 \geq -8x+9x-2$$

$$x+1 \geq x-2$$

$$x+1 \geq x-2$$

$$-x \quad -x$$

subtract x from both sides

$$1 \geq -2$$

→ This statement is always true therefore the solution set is every then



Compound Inequalities

Compound inequalities are inequalities where more than one inequality exists.

Depending on the type of inequalities, compound inequalities may be combined by "AND" or "OR"

Eg: $2x-1 \leq 3x+5 \leq 2x+11$

2: $5 < x+2 < 12$

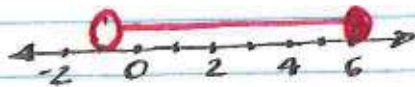
3: $-3 > 2x-1 \geq 4$

Solving compound inequalities

Eg 1: $-2 < x-1 \leq 5$

$+1 \quad +1 \quad +1$ add 1 to everything

$-1 < x \leq 6$



S.S. = $\{-1 < x \leq 6\}$

Alternative method
split the compound inequality

$-2 < x-1 \leq 5$

$-2 < x-1$ AND $x-1 \leq 5$

$+1 \quad +1$ AND $+1 \quad +1$

$-1 < x$ AND $x \leq 6$



The solution is the intersection of the two inequalities

S.S. = $\{-1 < x \text{ AND } x \leq 6\} = \{-1 < x \leq 6\}$

AND is equivalent to Intersection.
OR is equivalent to Union.

Ex 2: $3x-5 \leq 4$ OR $4-2x \geq 10$

$$\begin{array}{r} 3x-5 \leq 4 \\ +5 \quad +5 \end{array}$$

$$\frac{3x \leq 9}{3} \quad \frac{3}{3}$$

$$x \leq 3$$

$$\begin{array}{r} 4-2x \geq 10 \\ -4 \quad -4 \end{array}$$

$$\frac{-2x \geq -14}{-2} \quad \frac{-14}{-2}$$

$$x \leq 7$$

Remember to switch the inequality sign when dividing by negative #'s.



Since the compound inequalities are connected by "OR" the solution is going to be the "Union" of the two individual solutions.

$$S.S. = \{ x \leq 7 \}$$

OR

If you want to answer in interval notation

$$S.S. = \{ (-\infty, 7] \}$$

Relations and functions

A relation is defined as a set of ordered pairs.
A set is a collection of things.

Eg: a set is denoted by the symbol $\{ \}$

$$R = \{ (\text{Brother}, \text{sister}), (\text{Boy}, \text{girl}) \}$$

$$R = \{ (1, 2), (4, 5) \}$$

$$R = \{ (\text{Home}, \text{money}), (\text{food}, \text{money}), (\text{Love}, \text{money}) \}$$

notice that each relation contains an ordered pair/pairs.

Remark:

The first entry of an ordered pair is commonly known as an Input.

The second entry is known as an Output.

A function is a relation where each input is paired with exactly one distinct output.

Eg: 1

$$f = \{ (1, 2), (3, 4), (5, 6) \}$$

notice that the inputs of this relation

$$\{ 1, 3, 5 \}$$

each input is paired with exactly one unique output

1 paired with 2

3 paired with 4

5 paired with 6.

Therefore; f is a function.

Eg: 2

$$g = \{ (1, 2), (3, 4), (1, 5) \}$$

$$\text{input} = \{ 1, 3 \}$$

$$\text{output} = \{ 2, 4, 5 \}$$

1 is paired with 2

1 is also paired with 5

3 is paired with 4

Since 1 is paired with two different outputs, g is NOT a function.

More Examples:

3. $f = \{(1,4), (-2,3), (5,4)\}$

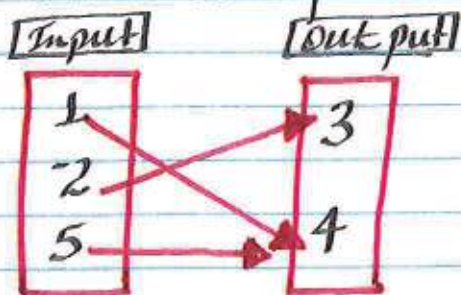
1 is paired with 4
-2 is paired with 3
5 is paired with 4

each input is paired with one unique output. Notice that 1 and 5 are paired with 4 and that is okay,

$\therefore f$ is a function!

This example shows that an output can have multiple input, but an input can not have more than one output & be a function.

Venn Diagram for Input / output
The above example can be represented by

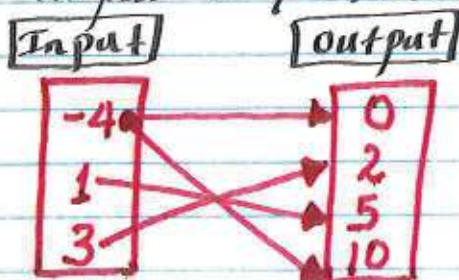


Function

4. $g = \{(-4,0), (1,5), (-4,10), (3,2)\}$

Input = $\{-4, 1, 3\}$

Output = $\{0, 2, 5, 10\}$

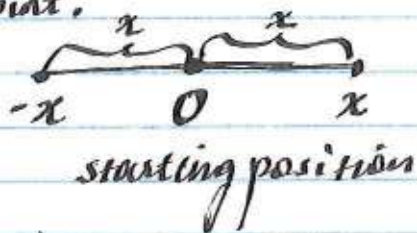


Since -4 is paired with two different outputs, (0 & 10), g is NOT a function.

Absolute Values.

Absolute values are represented by two long parallel lines "||".

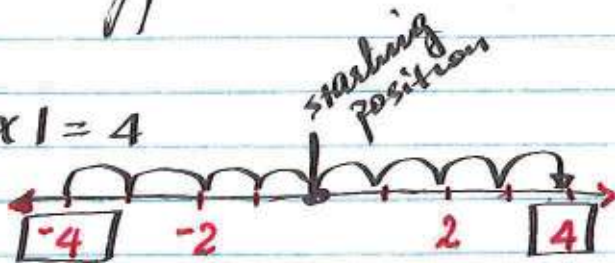
Absolute values produce positive output. It can be seen as the distance from a fixed point.



$|x|$ is always positive

Eg 1:

$$|x| = 4$$



Absolute value can also be defined as distance on a number line.

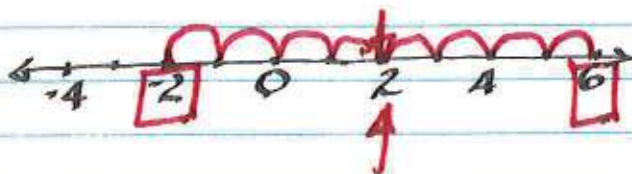
moving forward
moving backward } distance is always positive.

Eg 2: $|x - 2| = 4$

the starting position is found by setting whatever is inside the absolute value sign to 0 .

$$x - 2 = 0$$
$$+ 2 \quad + 2$$

$$x = 2 \quad \text{starting position}$$



$$s.o.s. = \{-2, 6\}$$

Remark:

Anything that comes out of an absolute value is always positive

Eg:

$$|4| = 4$$

$$|-x| = x$$

$$|-4| = 4$$

$$|-2| = 2$$

Solving absolute equations:

whenever you solve absolute value equations, you need to consider "TWO" cases:

1st case:

Treat the absolute value sign like a parenthesis and solve

2nd case:

• Change the signs of everything inside the absolute value sign and solve.

Eg 1. $|x+2| = 3$

case 1

$$|x+2| = 3$$

$$(x+2) = 3$$

$$x+2 = 3$$

$$\begin{array}{r} -2 \quad -2 \\ \hline \end{array}$$

$$x = 1$$

case 2

$$|x+2| = 3$$

$$(-x-2) = 3$$

$$-x-2 = 3$$

$$\begin{array}{r} +2 \quad +2 \\ \hline \end{array}$$

$$-x = 5$$

$$\begin{array}{r} \overline{-} \quad \overline{-} \\ \hline \end{array}$$

$$x = -5$$

$$S \cdot S = \{-5, 1\}$$

check:

$$|x+2| = 3$$

$$|-5+2| = 3$$

$$|-3| = 3$$

$$3 = 3 \checkmark$$

$$|x+2| = 3$$

$$|1+2| = 3$$

$$|3| = 3$$

$$3 = 3 \checkmark$$

MORE Examples:

2. $2|x-3|-4=6$

Case 1:

$$2(x-3)-4=6$$

$$2x-6-4=6$$

$$2x-10=6$$

$$+10 \quad +10$$

$$\frac{2x}{2} = \frac{16}{2}$$

$$x=8$$

Case 2:

$$2(-x+3)-4=6$$

$$-2x+6-4=6$$

$$-2x+2=6$$

$$-2 \quad -2$$

$$\frac{-2x}{-2} = \frac{4}{-2}$$

$$x=-2$$

check

$$2|x-3|-4=6$$

$$2|8-3|-4=6$$

$$2|5|-4=6$$

$$10-4=6$$

$$6=6 \checkmark$$

$$2|x-3|-4=6$$

$$2|-2-3|-4=6$$

$$2|-5|-4=6$$

$$2(5)-4=6$$

$$10-4=6$$

$$6=6 \checkmark$$

$$S.S. = \{x = -2, 8\}$$

3. $\frac{1}{2}|2x+1|+2=5$

Case 1:

$$\frac{1}{2}(2x+1)+2=5$$

$$x+\frac{1}{2}+2=5$$

$$x+\frac{5}{2}=5$$

$$-\frac{5}{2} \quad -\frac{5}{2}$$

$$x=5-\frac{5}{2}$$

$$= \frac{10}{2} - \frac{5}{2}$$

$$= \frac{10-5}{2}$$

$$x = \frac{5}{2}$$

Case 2:

$$\frac{1}{2}(-2x+1)+2=5$$

$$-x-\frac{1}{2}+2=5$$

$$-x+\frac{3}{2}=5$$

$$-\frac{3}{2} \quad -\frac{3}{2}$$

$$-x=5-\frac{3}{2}$$

$$= \frac{10}{2} - \frac{3}{2}$$

$$= \frac{10-3}{2}$$

$$-x = \frac{7}{2} \quad \times -1$$

$$x = -\frac{7}{2}$$

$$S.S. = \{x = -\frac{7}{2}, \frac{5}{2}\}$$

Eg 4:

$$2|2x-10| = -2$$

By just looking at the Right side of the equation, we can tell that there is NO solⁿ. This is because, no matter what x we can find, when we put it in the left side, it will always be positive and that doesn't equal -2 .

Let's solve

Case 1:

$$2(2x-10) = -2$$

$$4x - 20 = -2$$

$$+20 \quad +20$$

$$\frac{4x}{4} = \frac{18}{4}$$

$$x = 4.5$$

Case 2:

$$2(-2x+10) = -2$$

$$-4x + 20 = -2$$

$$-20 \quad -20$$

$$\frac{-4x}{-4} = \frac{-22}{-4}$$

$$x = 5.5$$

Check:

$$2|2x-10| = -2$$

$$x = 4.5$$

$$2|2(4.5)-10| = -2$$

$$2|9-10| = -2$$

$$2|-1| = -2$$

$$2(1) = -2$$

$$2 = -2 \text{ False}$$

$$x = 5.5$$

$$2|2(5.5)-10| = -2$$

$$2|11-10| = -2$$

$$2|1| = -2$$

$$2(1) = -2$$

$$2 = -2 \text{ False}$$

As a result the solution set is an Empty Set.

$$S.S. = \{ \emptyset \}$$

→ symbol for Empty set

Absolute Value Inequalities

An absolute value inequality is similar to an absolute value equation in many ways.

When we solve an inequality, we need to make sure that our solution set is a set of possible values that make the inequality true.

Important:

whenever we multiply or divide an inequality by a negative number, the inequality sign switches.

Eg 1:

$$-x > 5$$

$$\begin{array}{c} \times -1 \\ x < -5 \end{array}$$

$$x < -5$$

Eg 2:

$$-x \leq -2$$

$$\begin{array}{c} \div -1 \\ x \geq 2 \end{array}$$

$$x \geq 2$$

Notice that in both cases the inequality sign switches after multiplying or dividing by a negative number.

Solving absolute value Inequalities

Eg 1

$$|x-2| < 3$$

$$S.S = \{x: x < 5 \text{ And } x > -1\}$$



Case 1

$$(x-2) < 3$$

$$\begin{array}{c} x-2 < 3 \\ +2 \quad +2 \\ x < 5 \end{array}$$

$$x < 5$$

AND

Case 2

$$(-x+2) < 3$$

$$\begin{array}{c} -x+2 < 3 \\ -2 \quad -2 \\ -x < 1 \end{array}$$

$$\begin{array}{c} -x < 1 \\ \div -1 \quad \div -1 \\ x > -1 \end{array}$$

$$x > -1$$

More Examples:

Ex 2.

$$2|x-6| \leq 2$$

Case 1:

$$2(x-6) \leq 2$$

$$2x - 12 \leq 2$$

$$+12 \quad +12$$

$$\frac{2x}{2} \leq \frac{14}{2}$$

$$x \leq 7$$

Case 2:

$$2(-x+6) \leq 2$$

$$-2x + 12 \leq 2$$

$$-12 \quad -12$$

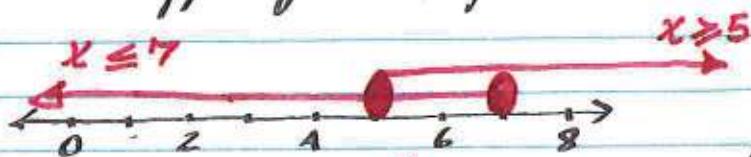
$$\frac{-2x}{-2} \leq \frac{-10}{-2}$$

$$x \geq 5$$

AND

* And implies that the solⁿ is an intersection of the two cases:

* on a number line, it is the overlapping set of points.



overlapping region

$$S.S. = \{x: x \geq 5 \text{ And } x \leq 7\}$$

OR

$$S.S. = \{x: 5 \leq x \leq 7\}$$

OR

$$S.S. = \{x: [5, 7]\}$$

Remark

It is always important to remember to check your solⁿ set.

* Some time, even if we can solve the inequalities, the solⁿ doesn't work.

check:

$$\text{let } x = 6$$

$$6 \text{ is b/n } 5 \text{ \& } 7$$

$$2|x-6| \leq 2$$

$$2|6-6| \leq 2$$

$$2|0| \leq 2$$

$$0 \leq 2 \quad \checkmark$$

eg 3:

Before we conclude that the solution set for a given absolute value inequality, it is very important to check the answers.

For example:

$$4|2x-10| < -8$$

Does NOT Have a solⁿ

$$4|2x-10| < -8$$

case 1

$$4(2x-10) < -8$$

$$8x-40 < -8$$

$$+40 \quad +40$$

$$8x < 32$$

$$\frac{8x}{8} < \frac{32}{8}$$

$$x < 4$$

case 2

$$4(-2x+10) < -8$$

$$-8x+40 < -8$$

$$-40 \quad -40$$

$$-8x < -48$$

$$\frac{-8x}{-8} < \frac{-48}{-8}$$

$$x > 6$$

$$S.S = \{ \emptyset \}$$



from the graph, we can see that the two cases don't have any common intersection.

As a result there is no solⁿ for the above absolute value inequality.

$$4|2(5)-10| < -8$$

$$4|0| < -8$$

$$0 < -8 \text{ False}$$

Check for

$$x=0 \quad x=6 \quad x=5$$

$$4|2(0)-10| < -8$$

$$4|2(6)-10| < -8$$

$$4|0-10| < -8$$

$$4|-10| < -8$$

$$4(10) < -8$$

$$40 < -8 \text{ False}$$

$$4|2(6)-10| < -8$$

$$4|2-10| < -8$$

$$4|2| < -8$$

$$8 < -8 \text{ False}$$

Ratio

A ratio is a comparison between two numbers by division.

Notation: $\frac{a}{b}$

For example

- $4:5$ Read as four to five.
- $\frac{4}{5}$ " " " " " .

A Unit rate:

A unit rate is when we compare a number with 1.

i.e. $\frac{12}{3}$ is NOT a unit rate: This is because 12 is compared with 3 not 1.

If we want to change $\frac{12}{3}$ into a unit rate, we need to divide 12 by 3.

$$\frac{12}{3} = \frac{4}{1} \quad \text{Now } \frac{4}{1} \text{ is a unit rate}$$

Change $\frac{5}{2}$ into a unit rate:

$$\frac{5}{2} = 5 \div 2 = \frac{2.5}{1} \quad \text{unit rate } \frac{2.5}{1}$$

Eq. Let 4 oz of tomatoes cost 5 \$
what is the unit rate?

$$\frac{4 \text{ oz}}{5 \$} = 4 \div 5 = \frac{0.8 \text{ oz}}{1 \$}$$

$$= \frac{0.8 \text{ oz}}{1 \$}$$

$$= 0.8 \text{ oz} / \$$$

The unit rate implies that for every dollar, we spent, we get 0.8 oz tomatoes.

Working with proportions

A proportion is comparison between two ratios by the equal sign

Eq: $\frac{1}{2} = \frac{4}{8}$ read as:
one half is proportional to 4, 8th.

In general a proportion is represented

by: $\frac{a}{b} = \frac{c}{d}$ where $a, b, c, d \in \mathbb{Z}$ and $b \neq 0$, and $d \neq 0$

a and d are extremes of the proportion &
 b and c are means of the proportion.

To solve proportions, we can use cross product and multiplication property of equality.

Eq: $\frac{x}{4} = \frac{2}{8}$

$$\frac{x}{4} = \frac{2}{8}$$

$$(x)(8) = (4)(2)$$

$$\frac{8x}{8} = \frac{8}{8}$$

$$\boxed{x = 1}$$

Eq: $\frac{x+2}{3} = \frac{6}{2}$

$$(2)(x+2) = (3)(6)$$

$$2x + 4 = 18$$

$$-4 \quad -4$$

$$\frac{2x}{2} = \frac{14}{2}$$

$$\boxed{x = 7}$$

cross product:
extremes = means
product product

check

$$\frac{x+2}{3} = \frac{6}{2}$$

$$\frac{7+2}{3} = \frac{6}{2}$$

$$\frac{9}{3} = \frac{6}{2}$$

$$3 = 3 \checkmark$$

Eg

Solve the proportion

$$\frac{2x+1}{4} = \frac{x+5}{2}$$

extremes = means
product product

$$(2)(2x+1) = (4)(x+5)$$

$$4x+2 = 4x+20$$

-2 -2

$$4x = 4x+18$$

-4x -4x

$$0 = 18 \text{ False}$$

Therefore; there is no such 'x' that makes the proportion true:

No solution

Eg solve the proportion

$$\frac{6}{3x+6} = \frac{2}{x+2}$$

$$(6)(x+2) = (2)(3x+6)$$

$$6x+12 = 6x+12$$

$$-12 \quad -12$$

$$6x = 6x+0$$

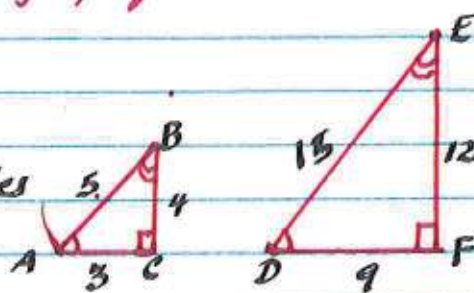
$$-6x \quad -6x$$

$$0 = 0 \text{ True}$$

This implies that the proportion is an Identity.
Therefore; any 'x' value makes the proportion true $x \in \mathbb{R}$.

Similar Triangles and Application of proportions

Similar triangles have the same angles and proportional dimensions / sides



Symbol for similarity (\sim)

$$\triangle ABC \sim \triangle DEF$$

(\cong) congruent

because $\angle A \cong \angle D$ and $\frac{DF}{AC} = \frac{9}{3} = 3$
 $\angle B \cong \angle E$
 $\angle C \cong \angle F$ $\frac{EF}{BC} = \frac{12}{4} = 3$

$$\frac{DE}{AB} = \frac{15}{5} = 3$$

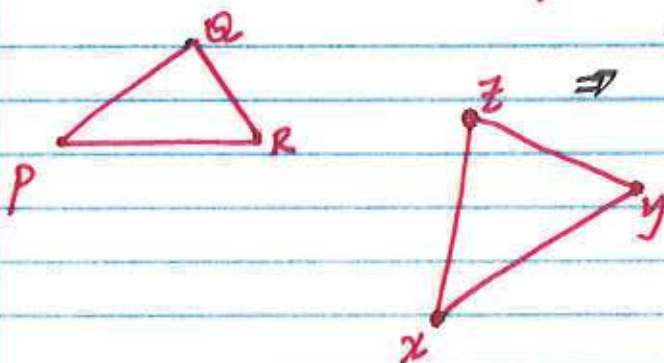
All the angles are congruent and the sides are proportional.

$$\Rightarrow \triangle ABC \sim \triangle DEF$$

It is very important to write the similarity statement in the right order.

i.e. $\angle A$ must match with $\angle D$
 $\angle B$ " " " $\angle E$
 $\angle C$ " " " $\angle F$

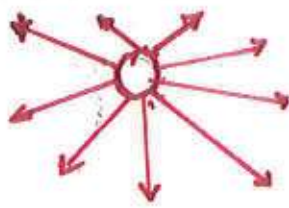
Let $\triangle XYZ \sim \triangle PQR$



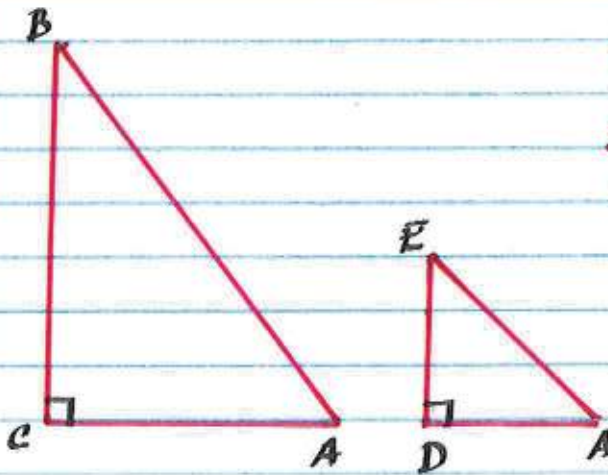
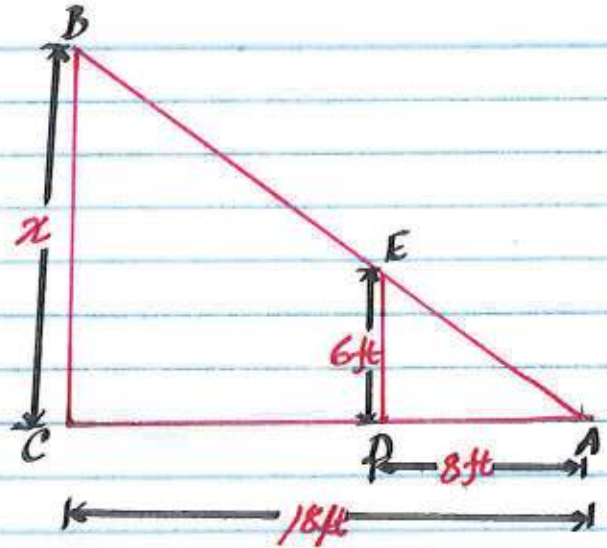
$$\angle X \cong \angle P \quad \angle Y \cong \angle Q \quad \angle Z \cong \angle R$$

$$\frac{XY}{PQ} = \frac{YZ}{QR} = \frac{XZ}{PR}$$

$$\triangle XYZ \sim \triangle PQR$$



A 6ft tall man is standing in front of a tree. If the sun casts an 8ft shadow and the tree casts 18ft shadow, how tall is the tree. (Assume that the two shadows ends at the same point)



From the figure, we can create two triangles using the tree and the man together with their shadows.

$$\triangle AED \sim \triangle ABC$$

$$\Rightarrow \frac{AE}{AB} = \frac{AD}{AC} = \frac{ED}{EC}$$

Since \overline{AE} & \overline{AB} are not given dimension, we exclude them from the proportion and use only $\frac{AD}{AC} = \frac{ED}{EC}$

Substituting their values

$$\frac{AD}{AC} = \frac{ED}{EC}$$

$$\frac{8ft}{18ft} = \frac{6ft}{x}$$

$$\frac{x(8ft)}{8ft} = \frac{(18 \times 6)ft^2}{8ft}$$

$$x = 13.5ft$$

The height of the tree is 13.5ft.

A percent is a portion of something out of a 100.

normally percents are written followed by '%' symbol.

Eg:

20%, 5%, 2.5%, 15%, 250%

Notice that a percent could be more than a 100, but we don't see it regularly.

Percents are very useful in comparing unrelated items.

for example two students have \$100 each. If each student spend \$25 to buy a book, then both student spend 25% of their money.

what if the 1st student has only \$50 and still spent \$25 to buy the book?

In this case the 1st student spend 50% of his/her money.

By just looking the money they spent, we can not really compare whom spent the most money based on what they had.

There are three common variables associated with percent.

1. percent (P)
2. percent amount (PA)
3. Total amount (TA)

a percent to a 100 is proportional to PA to TA.

$$\frac{P}{100} = \frac{PA}{TA}$$

Using the above formula, we can answer questions like

1. what is 10% of 60?
2. 6 is what percent of 60?
3. 6 is 10% of what?

The 1st question is asking the percent amount (PA)

The 2nd question is asking the percent (P)

The 3rd question is asking the total amount (TA)

we can use cross product to find all of the above questions. i.e.

$$\frac{P}{100} = \frac{PA}{TA}$$

$$(10)(60) = x(100)$$

$$\frac{600}{100} = \frac{100x}{100}$$

$$\frac{10}{100} = \frac{x}{60}$$

$$6 = x$$

Examples

1. 15 is what percent of 45?

Required: percent (p)

$$\frac{P}{100} = \frac{PA}{TA}$$

$$\frac{x}{100} = \frac{15}{45}$$

$$(45)(x) = (15)(100)$$

$$\frac{45x}{45} = \frac{1500}{45}$$

$$x = 33.3\%$$

2. 12 is 30% of what?

Required: Total amount (TA)

$$\frac{P}{100} = \frac{PA}{TA} ; \quad \frac{30}{100} = \frac{12}{x}$$

$$30x = (12)(100)$$

$$\frac{30x}{30} = \frac{1200}{30}$$

$$x = 40$$

3. What is 150% of 40?

Required: percent amount (PA)

$$\frac{P}{100} = \frac{PA}{TA} ; \quad \frac{150}{100} = \frac{x}{40}$$

$$(150)(40) = (100)(x)$$

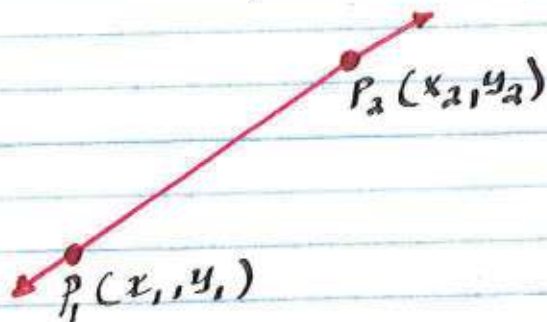
$$\frac{6000}{100} = \frac{100x}{100}$$

$$60 = x$$

Linear Equations

There is only one unique line that passes through two points.

The graph of a linear equation is a line, more specifically a straight line



A point can be represented by a coordinate. The 1st entry of a coordinate is the x-value and the second entry is called the y-value.

The slope of a line is the rate at which the y-value changes with respect to x-value change. In short

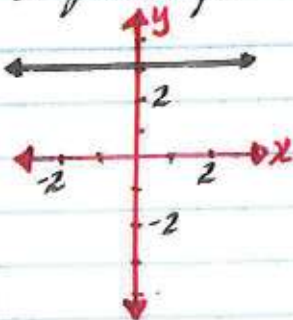
slope is defined as rise over run.

Rise - the vertical change b/w the y coordinates

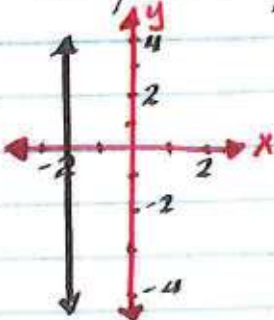
Run - the horizontal change b/w the x coordinates.

There are four different types of slopes.

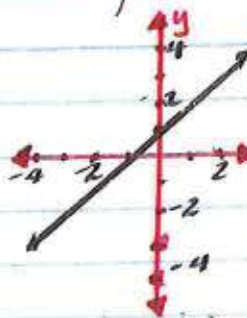
a zero slope



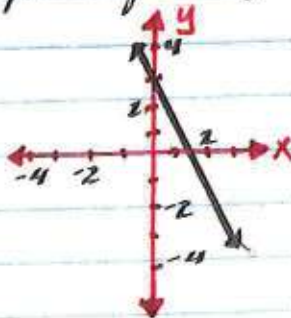
undefined slope

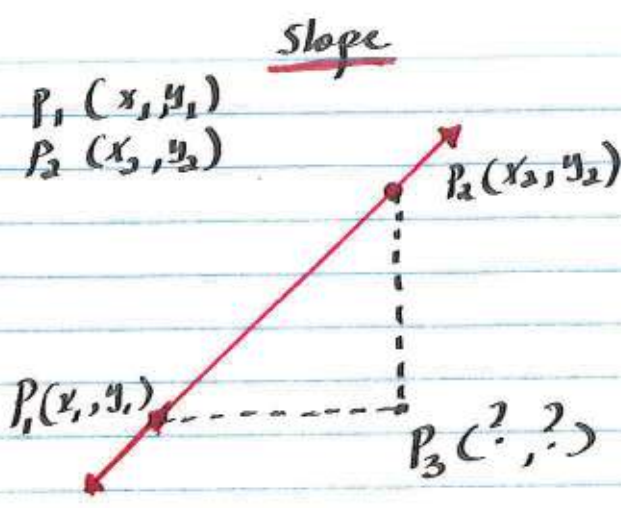


positive slope



negative slope





Remark:
 The x values remains constant in any vertical lines.

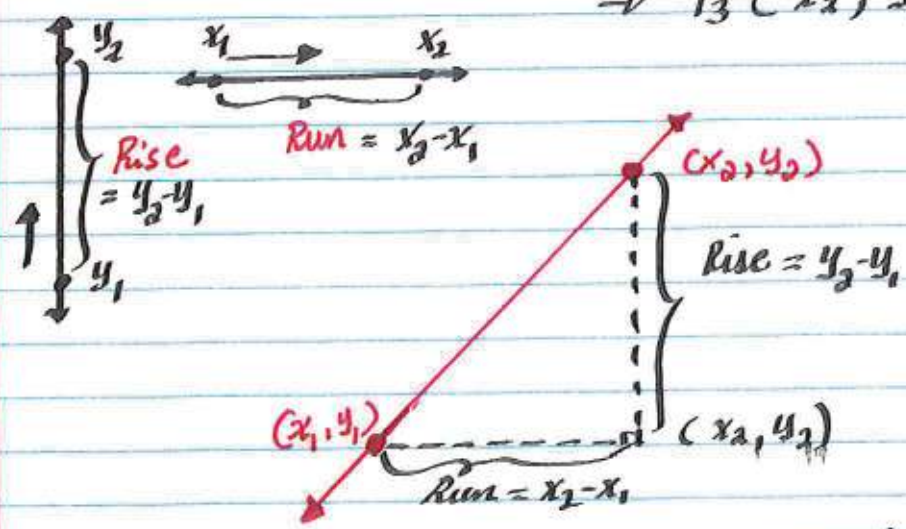
The y values remains constant in any horizontal lines

The coordinates of P_3 are unknown at the moment. But we can find them by applying the coordinates of horizontal and vertical lines properties.

the y coordinate of P_3 lies on the same horizontal line as P_1 and

The x coordinate of P_3 lies on the same vertical line as P_2 .

$\Rightarrow P_3(x_2, y_1)$



Slope = $\frac{\text{Rise}}{\text{Run}}$
 = $\frac{y_2 - y_1}{x_2 - x_1}$

Slope is represented by the letter 'm'

$m = \frac{y_2 - y_1}{x_2 - x_1}$

* Notice that the slope of a horizontal line is 0. This is because there is 'no' rise in a horizontal line

$$m = \frac{0}{x_2 - x_1}, \quad x_2 - x_1 \neq 0$$

$$= 0$$



$$m = \frac{y_1 - y_1}{x_2 - x_1} = \frac{0}{x_2 - x_1}$$

$$m = 0$$

* Notice that the slope of a vertical line is undefined. This is because there is no run implying that $\text{run} = 0$

$$m = \frac{y_2 - y_1}{0}$$

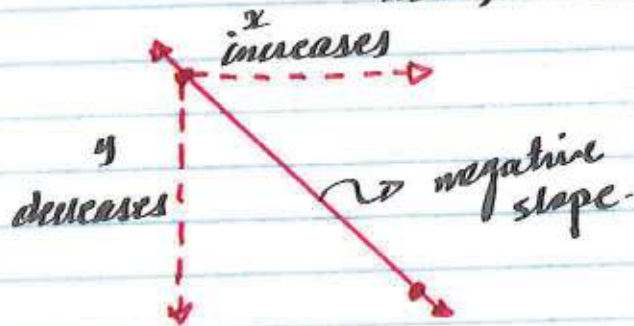
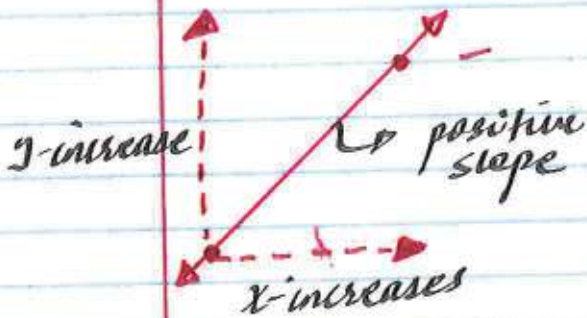
Dividing by zero is not possible!

(x_2, y_2)

(x_2, y_1)

$$m = \frac{y_2 - y_1}{x_2 - x_2} = \frac{y_2 - y_1}{0}$$

$$m = \text{A}$$



As the slope of a line gets bigger and bigger, the line gets steeper and steeper

As the slope of a line approaches to zero, the line gets flatter and flatter.

Examples:

Eg 1. Find the slope of a line that passes through

A (2, 4) B (8, 4)

let A (2, 4) B (8, 4)
 x_1, y_1 x_2, y_2

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{4 - 4}{8 - 2} = \frac{0}{6}$$

$$m = \underline{\underline{0}}$$



Eg 2. Find the slope of a line that passes through

C (-3, 5) D (3, 6)

let C (-3, 5) D (3, 6)
 x_1, y_1 x_2, y_2

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{6 - 5}{3 - (-3)}$$
$$= \frac{1}{3 + 3} = \frac{1}{6}$$

$$m = \frac{1}{6}$$



Eg 3. Find the slope of a line that passes through

A (-2, 10) B (1, 3)

let A (-2, 10) B (1, 3)
 x_1, y_1 x_2, y_2

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - 10}{1 - (-2)}$$
$$= \frac{-7}{3}$$

$$m = -\frac{7}{3}$$



Eg 4:

Find the slope of a line that passes through

G (3, 8) D (3, -2)

let G (3, 8) D (3, -2)
 x_1, y_1 x_2, y_2

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{-2 - 8}{3 - 3}$$

$$= \frac{-10}{0} \quad \#$$

$$m = \#$$

Equation of a line

There are three different ways a line can be written.

1. Standard form
2. point slope form
3. slope intercept form

Let m represents the slope of a line and b represents the y -intercept.

Slope intercept form:

$$y = mx + b$$

Standard form

$$Ax + By = C \quad , a, b \text{ and } c \text{ are integer coefficients}$$

Point slope form

$$y - y_1 = m(x - x_1) \quad m - \text{slope}$$

(x_1, y_1) - a point on the line.

Notice that each form of a line relates x and y .

The graph of a linear equation is a straight line



Special Cases:

Horizontal lines.

A horizontal line has a zero slope and a constant y value.

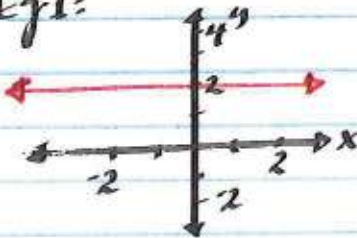


The equation of a horizontal line is always given by the form

$$y = k$$

where k is the y value, which the line passes through.

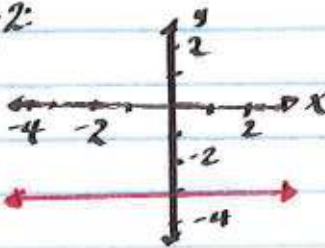
Eg 1:



The equation of the line is

$y = 2$, since the line passes through 2 on the y -axis.

Eg 2:



The equation of the line

$y = k$ since the line passes through -3 on the y -axis
 $y = -3$

Vertical lines

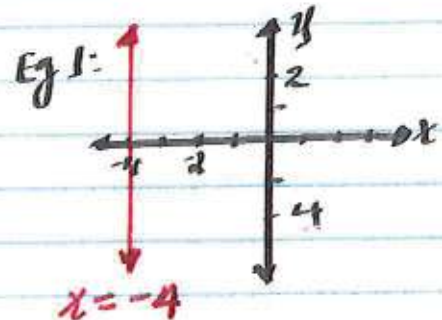


A vertical line has undefined slope & a constant x value.

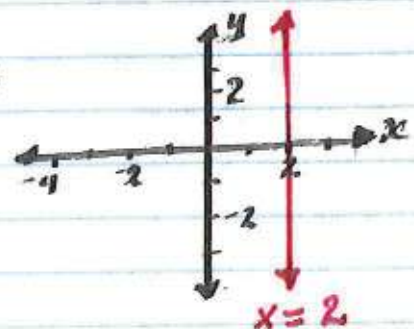
The equation of a vertical line is given by the form,

$$x = c$$

where c is the x value, which the line passes through.



Eg 2:



Techniques on how to write equations of a line:

Depending on the given information, we can write the equation of a line in three forms.

Question:

which equation form can we start?

Answers:

- If a slope and a point is given: slope-point form
- If a slope and y-intercept is given: slope-intercept
- If two points are given: point-slope form:

Eg 1:

Find the equation of a line that has slope 2 and y-intercept 5.

given: $m = 2$
 $b = 5$

$$y = mx + b$$

substitute m & b values

form: slope-intercept:

$$y = 2x + 5$$

Eg 2:

Find the equation of a line that has slope $\frac{1}{2}$ and passes through the point $(-4, 12)$.

given: $m = \frac{1}{2}$
 $(x_1, y_1) = (-4, 12)$

$$y - y_1 = m(x - x_1)$$

substitute m & (x_1, y_1)

form: point-slope:

$$y - 12 = \frac{1}{2}(x + 4)$$

Eg 3: Find the equation of a line that passes through $(3, 1)$ & $(-4, 15)$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{15 - 1}{-4 - 3} = \frac{14}{-7} = -2$$

given: $(3, 1)$ & $(-4, 15)$
 x_1, y_1 x_2, y_2

$$y - y_1 = m(x - x_1)$$

$$y - 1 = -2(x - 3) \quad \text{OR}$$

$$y - 15 = -2(x + 4)$$

first find the slope
form: point-slope:

THREE FORMS of a LINE:

In this section we will learn how to write an equation of a line in all the three forms:

steps:

- 1st: Determine which form to start the equation of a line depending on the given information.
- 2nd: simplify the 1st equation to the desired form
- 3rd: make sure that standard forms of a linear equation has integer coefficients.

Eg: write the equation of a line that passes through (4,5) & (2,-6) in all the three forms.

1st: let's find the slope:
given: (4,5) & (2,-6)
label $\rightarrow x_1, y_1, x_2, y_2$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-6 - 5}{2 - 4} = \frac{-11}{-2} = \frac{11}{2}$$

$$m = \frac{11}{2}$$

Now let's use the slope we calculated and the 1st point

form: point-slope form:

$$y - y_1 = m(x - x_1)$$

$$y - 5 = \frac{11}{2}(x - 4)$$

To find the slope intercept form:

$$y - 5 = \frac{11}{2}(x - 4)$$

$$y - 5 = \frac{11}{2}x - \frac{44}{2}$$

$$y - 5 = \frac{11}{2}x - 22$$

$$y = \frac{11}{2}x - 17$$

To find the standard form:

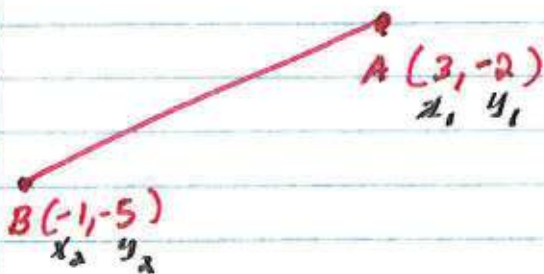
$$y = \frac{11}{2}x - 17$$

$$-\frac{11}{2}x + y = -17$$

$$-\frac{11}{2}x + y = -17$$

$$-11x + 2y = -34$$

Ex. Find the slope of a line that passes through points $A(3, -2)$ & $B(-1, -5)$



$$\begin{aligned}
 m &= \frac{y_2 - y_1}{x_2 - x_1} \\
 &= \frac{-5 - (-2)}{-1 - 3} \\
 &= \frac{-5 + 2}{-4} \\
 &= \frac{-3}{-4} \\
 &= \frac{3}{4}
 \end{aligned}$$

Ex. Write the equation of a line that has the same slope as the previous example ($\frac{3}{4}$) and has y-int 5.

Given:

$$y\text{-int} = 5 = b$$

$$m = \frac{3}{4}$$

Let's start with slope intercept

form: $y = mx + b$

$$y = \frac{3}{4}x + 5$$

standard form:

$$y = \frac{3}{4}x + 5$$

$$-\frac{3}{4}x - \frac{3}{4}x$$

$$-\frac{3}{4}x + y = 5$$

$$\times 4 \quad \times 4 \quad \times 4$$

$$-3x + 4y = 20$$

point slope form:
we have slope

$$m = \frac{3}{4} \text{ but}$$

we need to find a point x_1, y_1

notice: $P(4, 8)$ is on the line:

$$y - y_1 = m(x - x_1)$$

$$y - 8 = \frac{3}{4}(x - 4)$$

Eg: write the equation of a line with slope $-\frac{1}{4}$ and has a point $(2, 6)$; in all the three forms:

Given:

$$m = -\frac{1}{4}$$

$$P = (x_1, y_1) = (2, 6)$$

Start with:

point-slope form:

$$y - y_1 = m(x - x_1)$$

$$y - 6 = -\frac{1}{4}(x - 2)$$

To find slope-intercept distribute $-\frac{1}{4}$ over $(x-2)$ & simplify.

$$y - 6 = -\frac{1}{4}(x - 2)$$

$$y - 6 = -\frac{1}{4}x + \frac{1}{4}(2), \quad \frac{1}{4} \cdot 2 = \frac{2}{4}$$

$$y - 6 = -\frac{1}{4}x + \frac{1}{2} = \frac{1}{2}$$

$$+6 \quad +6, \quad \frac{1}{2} + \frac{6}{1}$$

$$y = -\frac{1}{4}x + \frac{13}{2}$$

$$\frac{1}{2} + \frac{12}{2}$$

$$\frac{13}{2}$$

To find standard form: Subtract $(-\frac{1}{4}x)$ from both sides & multiply everything by 4.

$$y = -\frac{1}{4}x + \frac{13}{2}$$

$$+\frac{1}{4}x \quad +\frac{1}{4}x$$

$$\frac{1}{4}x + y = \frac{13}{2}$$

$$\times 4 \quad \times 4 \quad \times 4$$

$$x + 4y = 26$$

✱

Notice that once we find point-slope form of line, it's a matter of simplifying the equation to get the other forms.

Horizontal & Vertical equations of a line

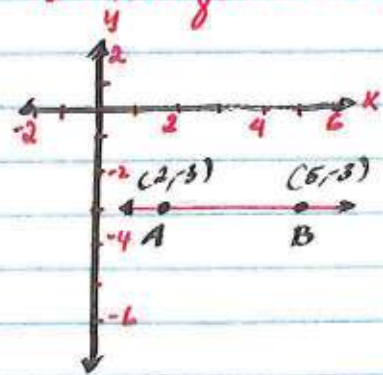
Recall from earlier Section: The equation of a horizontal & a vertical line is given by $y = k$ & $x = a$ respectively where k & a are constants.

How can we tell if a line that passes through two points is a horizontal or a vertical line?



If the points have the same x -coordinate, then the line is a vertical line.
If the two points have the same y -coordinate, then the line is a horizontal line.

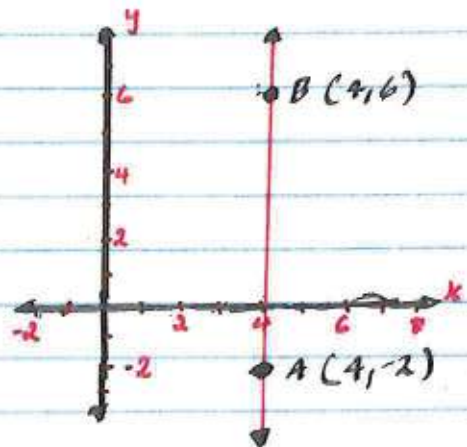
Eg: Find the equation of a line that passes through points $A(2, -3)$ & $B(5, -3)$



Notice that both points have the same y -coordinate. This implies that the line is a horizontal line

$$y = -3$$

Eg. Find the equation of a line that passes through points $A(4, -2)$ & $B(4, 6)$



Notice that both points have the same x -coordinates. This implies that the line is vertical

$$x = 4$$

Parallel And Perpendicular lines

Parallel lines are lines on a given plane that do not intersect at any point



Symbol (parallel " \parallel ")
 In the above figure
 l_1 is parallel to l_2
 $\Rightarrow l_1 \parallel l_2$

If two lines are parallel then they must have the same "slope".

Remark: parallel lines can not have the same slope and the same y-int.

If this happens, the lines are exactly the same.

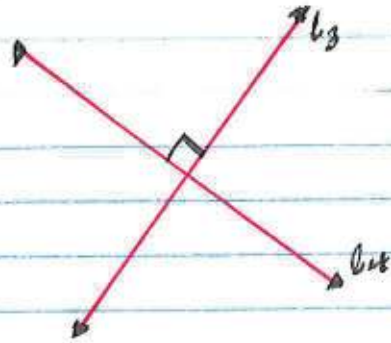
Eg: $l_1: y = 7x + 2$
 $l_2: y = 7x - 5$

let m_1 be the slope of l_1 & m_2 be the slope of l_2

$m_1 = 7$ since the y-int are different and $m_1 = m_2$
 $m_2 = 7$

$l_1 \parallel l_2$

Perpendicular lines are lines on a given plane that intersect at 90° .



Symbol (perpendicular " \perp ")
 In the above figure
 l_3 is perpendicular to l_4
 $\Rightarrow l_3 \perp l_4$

If two lines are perpendicular, then the "product" of their "slope" must be "-1"

Eg: $l_3: y = 2x - 6$
 $l_4: y = -\frac{1}{2}x + 3$

let m_1 be slope of l_3 & m_2 be slope of l_4 .

$m_1 = 2$

$m_2 = -\frac{1}{2}$

Now the product of m_1 & m_2 is

$m_1 * m_2$

$2 * -\frac{1}{2}$

$\boxed{-1}$

$\Rightarrow \boxed{l_3 \perp l_4}$

Eg. Find an equation of a line that is parallel to $y = 3x - 2$ and passes through $(-2, 3)$

Given: Eq 1: $y = 3x - 2$
 $P(-2, 3)$

$\Rightarrow m = 3$

parallel lines have the same slope.

Since we have slope & a point, we can use point slope form of a line to find the equation.

$$y - y_1 = m(x - x_1) \quad m = 3$$

$$P(-2, 3)$$

$$x_1, y_1$$

$$y - 3 = 3(x - (-2))$$

$$y - 3 = 3x + 2$$

$$+3 \quad +3$$

$$y = 3x + 5$$

Eg. Find the equation of a line that is perpendicular to the line $2x + 3y = 10$ and passes through the point $(4, 1)$.

Given:

Eq 1: $2x + 3y = 10$
 $P(4, 1)$

we can find the slope of our line from the given equation by using properties of perpendicular lines. i.e. the product of their slope is -1 .

$2x + 3y = 10$ needs to be rewritten
 $-2x \quad -2x$ in $y = mx + b$ form

$$\frac{3y}{3} = \frac{-2x + 10}{3}$$

$$y = -\frac{2}{3}x + \frac{10}{3}$$

Now we have $m = \frac{3}{2}, P(4, 1)$

$$y - y_1 = m(x - x_1)$$

$$y - 1 = \frac{3}{2}(x - 4)$$

$$y - 1 = \frac{3}{2}x - 6$$

$$+1 \quad +1$$

$$y = \frac{3}{2}x - 5$$

Since the slope = $-\frac{2}{3}$, our slope is $\frac{3}{2}$.

Check: $-\frac{2}{3} \times \frac{3}{2} = -1$.

Eg:

Find an equation of a line that is perpendicular to the line $2x + y = 4$ and has the same y-intercept as the line $4x + 2y = -6$

Given: two equations:

Eg 1: $2x + y = 4$
(perpendicular to this line)

Eg 2: $4x + 2y = -6$
(has the same y-int as this line)

Strategy:

- * get the slope of our line from the 1st eq and get the y-int of our line from the second equation

1st we need to rewrite both equations in slope intercept form.

Eg 1:

$$2x + y = 4$$
$$\begin{array}{r} -2x \\ -2x \end{array}$$

$$y = -2x + 4$$

Eg 2:

$$4x + 2y = -6$$
$$\begin{array}{r} -4x \\ -4x \end{array}$$

$$\frac{2y}{2} = \frac{-4x - 6}{2}$$

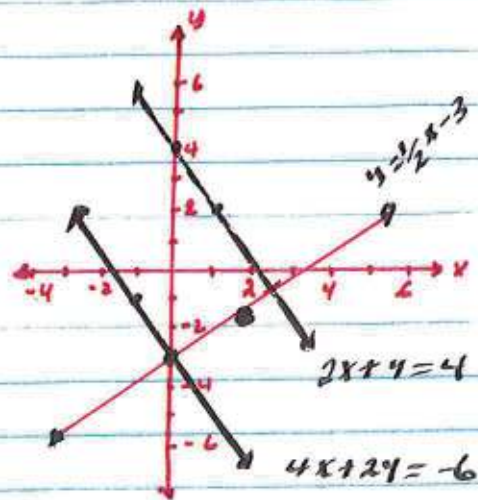
\Rightarrow our slope is $\frac{1}{2}$
since $-2 + \frac{1}{2} = -1$.

$$y = -2x - 3$$

\Rightarrow our y-int is -3
since they are the same.

Our equation $y = mx + b$ $m = \frac{1}{2}$
 $b = -3$

$$y = \frac{1}{2}x - 3$$



Systems of Linear Equations:

A system is when a couple groups of things get combined to create something.

In this specific case:

* System of linear equations is defined as two linear equations combined together.

Remark: There could be more than two linear equations, but for now we are only going to study two linear equations.

Notation:

we use a curly bracket to represent a system of linear equations.

" "

Eg: $\begin{cases} y = 2x + 1 \\ y = -x + 10 \end{cases}$

$\begin{cases} 3x - 2y = 5 \\ 2x + 1y = 4 \end{cases}$

$\begin{cases} \frac{1}{2}x - 3y = 6 \\ y = 2x + 2 \end{cases}$

Notice that each system of linear equations can be written as slope-intercept form, standard form or in both forms.

Solving a system of linear equations is basically find values of "x" & "y" that make both linear equations true.

For example $x = 1$ $y = 5$ (1, 5) is a solution for the system $\begin{cases} y = 2x + 3 \\ y = -3x + 8 \end{cases}$

" This is because if you put both $x = 1$ and $y = 5$ in each equation, you will get a true statement. "

Depending on the linear equations, a system of linear equations can have one of the following three solution types.

1. One solution
2. Many solutions
3. No solution

One solution

This type exists if and only if there is exactly "one" solution that makes both linear equations true.

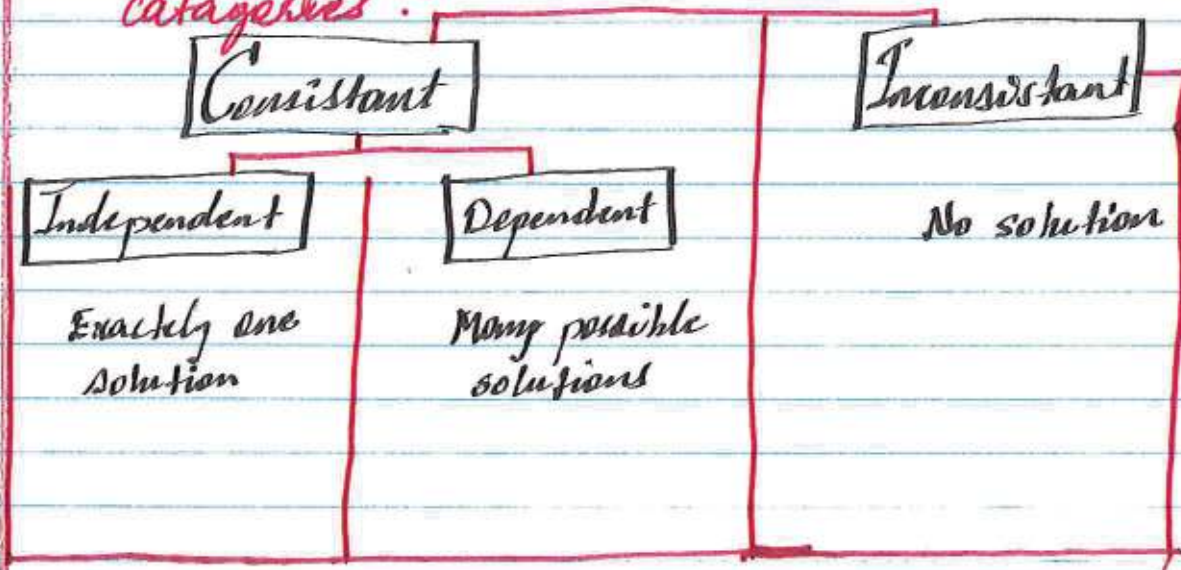
Many solutions

This type exists if there are many x and y that makes both linear equations true.

No solutions

This type exists if there are no such x & y that makes both equations true.

Based on the solution type we can classify systems of linear equations in two categories.



Systems of Linear Equation Solving Techniques.

There are three different techniques to solve systems of linear equations.

I Graphing

We can draw the graph of the two given equations and find the point where the two lines intersect.

II Substitution

We can rewrite one of the linear equations in slope intercept form and substitute 'y' in the other equation to solve for 'x' and then 'y'.

III Elimination

We can write both equations in standard form and eliminate either the x or y values to solve for the other.

These techniques make much more sense through examples.

Eg. which method would be more convenient to solve the following system of linear equations.

1.
$$\begin{cases} 2x - y = 8 \\ 2x + 4y = 6 \end{cases}$$

"Elimination"

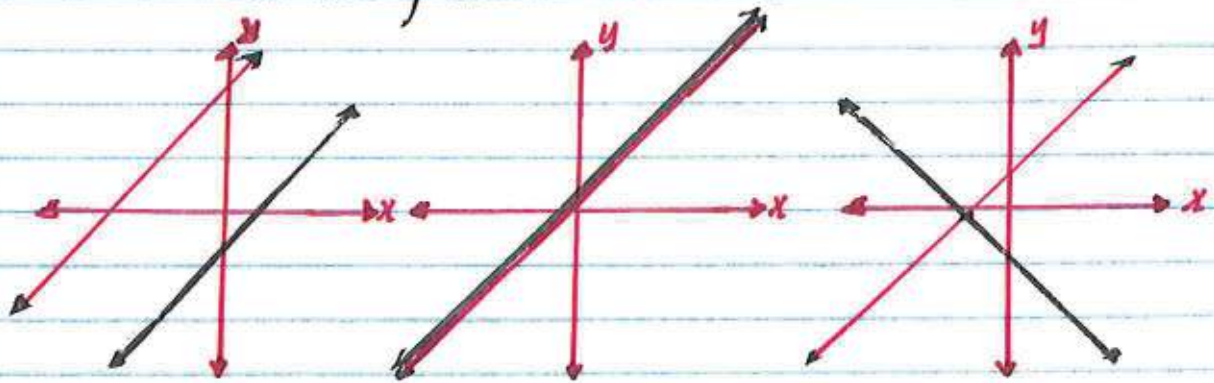
2.
$$\begin{cases} 2x + 3y = 12 \\ 3x + y = 6 \end{cases}$$

"Graphing"

3.
$$\begin{cases} y = 2x + 1 \\ 2x + y = 4 \end{cases}$$

"Substitution"

How to determine whether a given system of linear equations is consistent, inconsistent, dependent or independent.



parallel lines
"Inconsistent"

Same lines
"Consistent"
Dependent

intersecting lines
"Consistent"
Independent

→ If the two lines have the same slope but different y -intercepts, then the lines are parallel and hence not intersecting.

→ If the two lines have the same slope and identical y -intercepts, then the lines are exactly the same and they intersect every where.

consistent and independent solutions exist when the slope of the two equations are different.

Classify the following system of linear equations as consistent, inconsistent, dependent and independent.

Eg. 1.

$$\begin{cases} y = 3x + 5 \\ y = 2x - 1 \end{cases}$$

The slope of the 1st eq. is 3 and the 2nd eq. is 2. This implies that the system is consistent and independent.

Eg. 2.

$$\begin{cases} y = 3x - 5 \\ y = 3x + 2 \end{cases}$$

The slope of the 1st eq. is 3 and the 2nd eq. is 3. Since both equations have the same slope and different y-intercepts, the lines are parallel implying the system is inconsistent.

Eg. 3.

$$\begin{cases} y = \frac{1}{2}x + 3 \\ -6x + 12y = 36 \end{cases}$$

1st change the second equation into slope-intercept form

$$\begin{array}{r} -6x + 12y = 36 \\ +6x \qquad \qquad +6x \end{array}$$

$$\frac{12y}{12} = \frac{6x}{12} + \frac{36}{12}$$

$$y = \frac{1}{2}x + 3$$

Both equations have the same slope, i.e. ($\frac{1}{2}$) and y-intercept i.e. (3), the equations are exactly identical.

This implies, the system is consistent and dependent.

Ex.

Solve the following system of linear equation by graphing.

$$\begin{cases} 2x + 3y = 12 \\ 3x + y = 6 \end{cases}$$

Since both equations are given in standard form, we can easily find the x and y intercepts

Ex 1. $2x + 3y = 12$

x -int, plug $y=0$

$$2x + 3(0) = 12$$

$$\frac{2x}{2} = \frac{12}{2}$$

$$x = 6$$

y -int, plug $x=0$

$$2(0) + 3y = 12$$

$$\frac{3y}{3} = \frac{12}{3}$$

$$y = 4$$

x -int $(6, 0)$

y -int $(0, 4)$

Ex 2. $3x + y = 6$

x -int, plug $y=0$

$$3x + (0) = 6$$

$$\frac{3x}{3} = \frac{6}{3}$$

$$x = 2$$

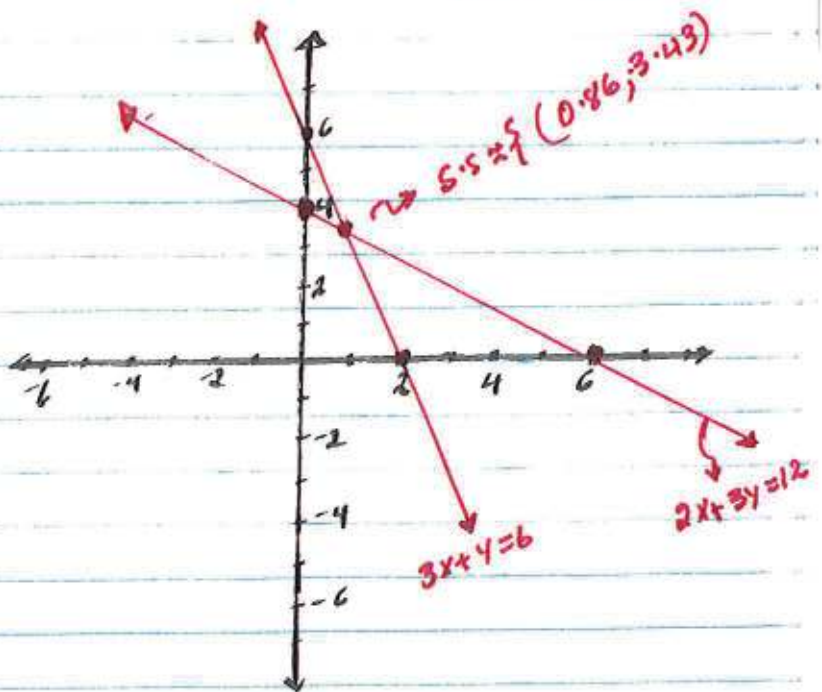
y -int, plug $x=0$

$$3(0) + y = 6$$

$$y = 6$$

x -int $(2, 0)$

y -int $(0, 6)$



$$S.S. = \left\{ \left(\frac{6}{7}, \frac{24}{7} \right) \right\}$$

Ex.

Solve the following systems of linear equations by substitution.

$$\begin{cases} y = 2x + 1 \\ 2x + y = 4 \end{cases}$$

Since the first equation is given in slope-intercept form, we can substitute it in the 2nd equation

$$y = 2x + 1$$

substitute the 'y' in the second equation by $2x + 1$

$$2x + y = 4$$

$$2x + (2x + 1) = 4$$

$$2x + 2x + 1 = 4$$

$$4x + 1 = 4$$

$$-1 = -1$$

$$4x = 3$$

$$\frac{4x}{4} = \frac{3}{4}$$

$$\boxed{x = \frac{3}{4}}$$

Now, since we know the value of 'x' we can put the 'x' value in the 1st equation to find 'y'.

$$y = 2x + 1$$

$$y = 2\left(\frac{3}{4}\right) + 1$$

$$y = \frac{6}{4} + 1$$

$$y = \frac{6}{4} + \frac{4}{4}$$

$$y = \frac{6+4}{4} = \frac{10}{4}$$

$$\boxed{y = \frac{5}{2}}$$

$$S.S. = \left\{ \left(\frac{3}{4}, \frac{5}{2} \right) \right\}$$

Ex. solve the following by elimination

$$\begin{cases} 2x - y = 8 \\ 2x + 4y = 6 \end{cases}$$

plug y in the 1st eq.

$$2x - \left(-\frac{2}{5}\right) = 8$$

$$2x + \frac{2}{5} = 8$$

$$2x = 8 - \frac{2}{5}$$

$$\frac{2x}{2} = \frac{38}{2} \quad \boxed{x = \frac{19}{1}}$$

subtract the 2nd equation from 1st

$$\begin{cases} 2x - y = 8 \\ - \quad 2x + 4y = 6 \end{cases}$$

$$-y - 4y = 8 - 6$$

$$-5y = 2$$

$$\frac{-5y}{-5} = \frac{2}{-5}$$

$$\boxed{y = -\frac{2}{5}}$$

$$S.S. = \left\{ \left(\frac{19}{1}, -\frac{2}{5} \right) \right\}$$

Example:

Determine whether the following system has a consistent or inconsistent solution and find the solution by using all the three techniques i.e. graphing, substitution & Elimination

$$\begin{cases} y = 2x + 2 \\ 2x + 3y = 6 \end{cases}$$

Let m_1 be slope of eq 1 and m_2 be eq 2.

$m_1 = 2$, to find the m_2 , we need to rewrite the 2nd eq in slope-int form:

$$\begin{aligned} 2x + 3y &= 6 \\ -2x \quad -2x \\ \hline 3y &= -2x + 6 \\ \frac{3y}{3} &= \frac{-2x}{3} + \frac{6}{3} \end{aligned}$$

$$y = \frac{-2}{3}x + 2$$

$$\Rightarrow m_2 = \frac{-2}{3}$$

Since the two equations have different slopes, the system is consistent and independent.

a) Substitution:

$$\begin{cases} y = 2x + 2 \\ 2x + 3y = 6 \end{cases}$$

$$\begin{aligned} 2x + 3(2x + 2) &= 6 & y &= 2x + 2 \\ 2x + 6x + 6 &= 6 & y &= 2(0) + 2 \\ 8x + 6 &= 6 & y &= 2 \\ -6 & -6 & & \\ \hline 8x &= 0 & & \\ x &= 0 & & \end{aligned}$$

$$S.S = \{(0, 2)\}$$

b) Elimination

$$\begin{aligned} y &= 2x + 2 \\ -2x \quad -2x \\ \hline \end{aligned}$$

$$\begin{aligned} -2x + y &= 2 \\ + \quad 2x + 3y &= 6 \\ \hline \end{aligned}$$

$$\frac{4y}{4} = \frac{8}{4}$$

$$y = 2$$

$$y = 2x + 2$$

$$2 = 2x + 2$$

$$-2 \quad -2$$

$$2x = 0 \quad x = 0$$

Graphing

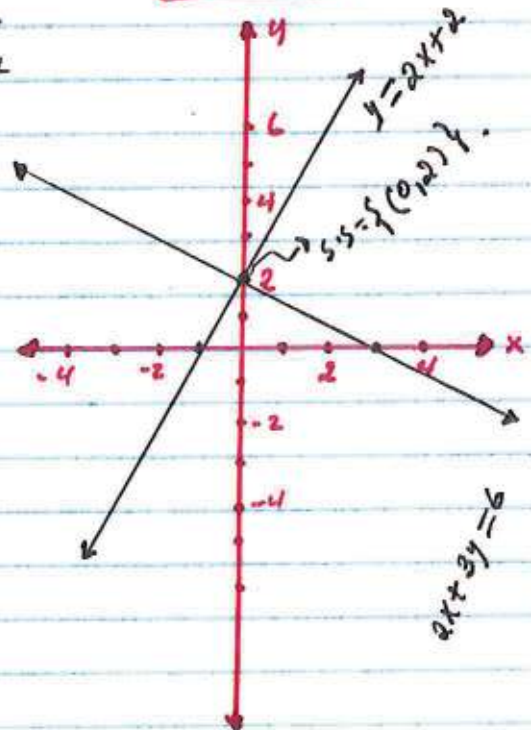
$$y = 2x + 2$$

0	2
-1	0

$$2x + 3y = 6$$

0	2
3	0

$$S.S = \{(0, 2)\}$$



Linear Inequalities

A linear inequality is an inequality with a linear function

From the definition a linear inequality has three components

1. has to be a function
2. has to be a linear
3. needs to have an inequality sign

Eq:

a) $y < 3x - 1$

b) $5x - 2y \geq 5$

c) $x > 4$

These are all linear inequalities

A solution of a linear inequality is a set of points (x, y) that make the inequality true.

Eq.

a) $y < 3x - 1$

$x = 4$
 $(4, 2)$ is a solution because

$$2 < 3(4) - 1$$

$$2 < 12 - 1$$

$$2 < 11 \text{ true}$$

unlike linear equations, linear inequalities have more than one solution

In the above example $(5, 1)$ is also a solution, because

$$1 < 3(5) - 1$$

$$1 < 15 - 1$$

$$1 < 14 \text{ true}$$

In order to find all the solutions of a given linear inequality, we need to draw its graph and shade the region of the solution set on the coordinate plane.

Technique for graphing and shading a linear inequality

1st. Make sure whether the line is broken or solid.

If the inequality sign is $<$ or $>$ Broken line $\cdots\cdots\cdots$
 \leq or \geq Solid line —————

Broken line means, the solution does not include all the points on the line and Solid lines means, the solution set includes the points on the line too.

2nd Draw the graph of the inequality. Find two points by changing the inequality into an equation but don't forget step 1 when you draw the line.

3rd Any line divides the coordinate plane into two parts. we just need to shade one of the two parts.

In order to decide which part to shade, choose a test point. Plug the test point into the inequality and see if it is true. If it is true shade the graph part that includes the test point. If not true shade the other part.

Example:

Find the solution set by graphing for $y < 3x - 1$

1st step:

Broken or solid line?

Since the inequality sign is ' $<$ ', the line is **broken**

2nd step:

Change the inequality in to an equation and find two points.

$$y = 3x - 1$$

x	y
0	-1
2	5

when $x = 0$

$$y = 3(0) - 1$$

$$y = -1$$

when $x = 2$

$$y = 3(2) - 1$$

$$= 6 - 1$$

$$y = 5$$

Recall that we can choose any 'x' values but the 'y' values depends on the 'x' values we choose.

Now we can draw the broken line through $(0, -1)$ & $(2, 5)$

3rd step:

Find any test point and check whether the inequality at that point is true or false.

Test point $(4, -2)$
x y

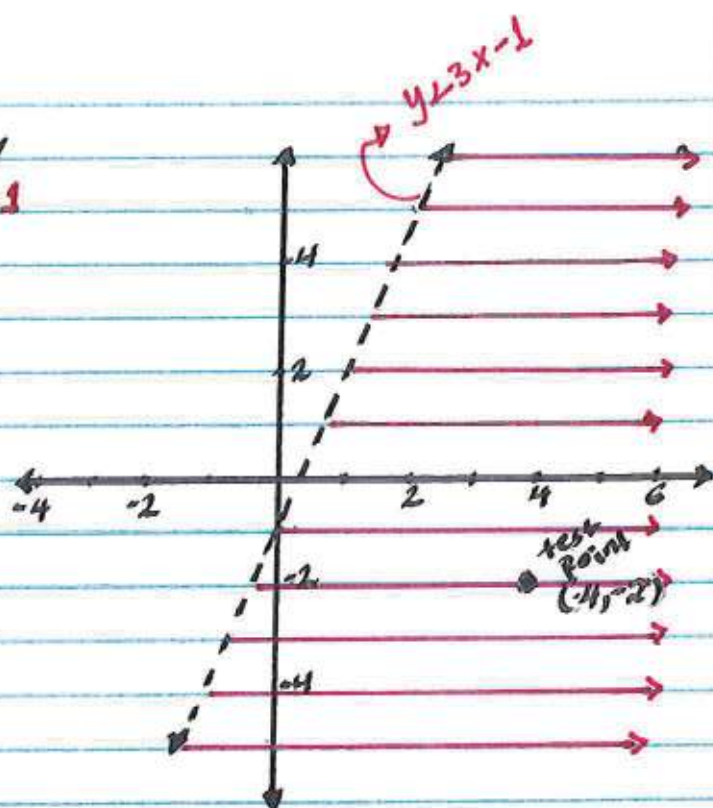
$$y < 3x - 1$$

$$-2 < 3(4) - 1$$

$$-2 < 12 - 1$$

$$-2 < 11 \quad \text{TRUE}$$

Therefore; we shade the graph towards the test point starting at the line.



Any point inside the shaded area/region is a solution

Ex: Find the solution for the inequality $3x + 4y \leq 12$

1st step:

Inequality sign \leq
as solid line.

2nd step:

points

x	y
0	3
4	0

$$3x + 4y = 12$$

$$x = 0$$

$$3(0) + 4y = 12$$

$$4y = 12$$

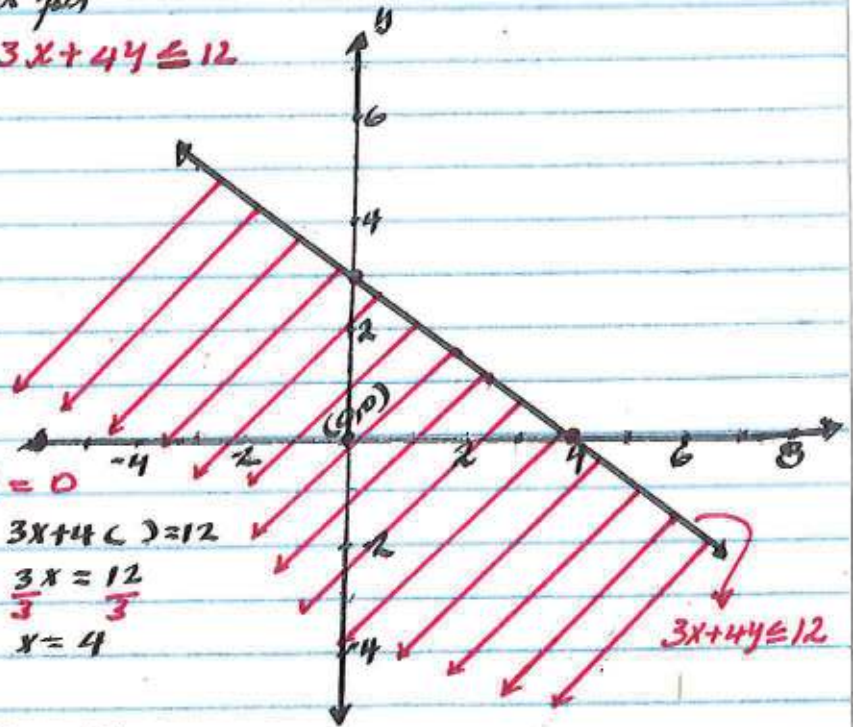
$$y = 3$$

$$y = 0$$

$$3x + 4(0) = 12$$

$$3x = 12$$

$$x = 4$$



3rd step:

Test point $(0, 0)$

$$3x + 4y \leq 12$$

$$3(0) + 4(0) \leq 12$$

$$0 \leq 12 \text{ True } \Rightarrow \text{shade towards } (0, 0)$$

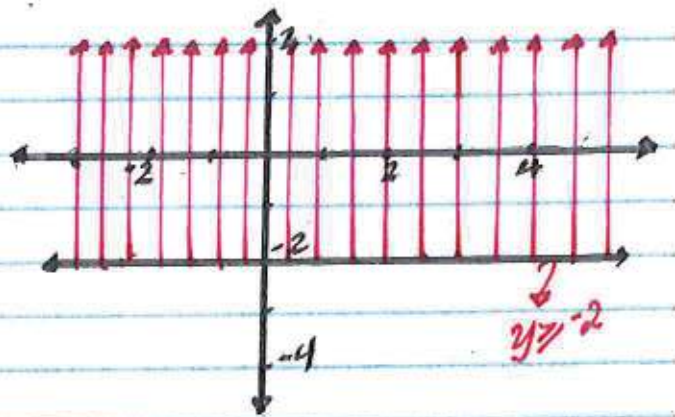
Ex: solution for $x < 3$

It is a vertical line that passes through $x=3$ and it has to be broken line



Ex: solution for $y \geq -2$

It is a horizontal line that passes through $y=-2$ and it needs to be a solid line.



Systems of Linear Inequalities.

A system of linear inequalities is a combination of two or more inequalities combined together.

In order to find a solution for a given system of linear inequalities, we follow the same procedures just like linear inequalities.

Graphically, the solution of system of linear inequalities is the intersection the individual shaded region.

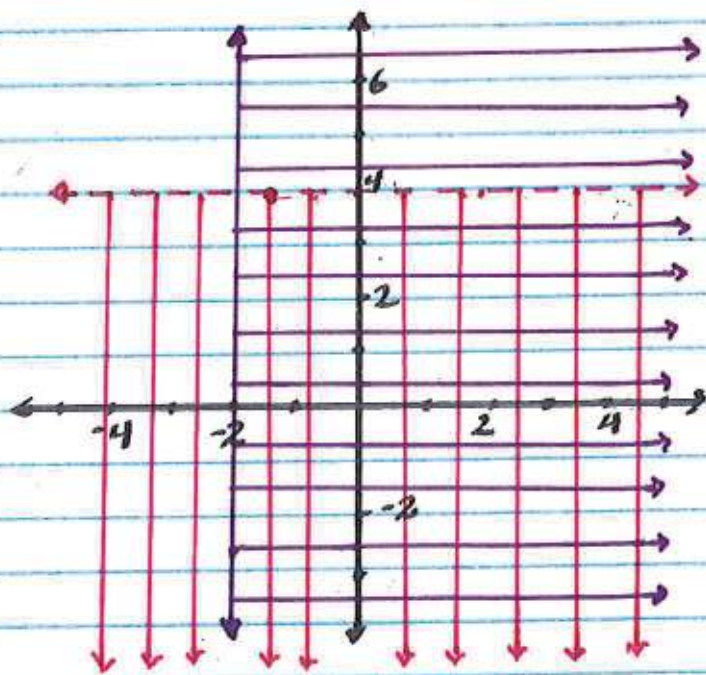
Eg:
$$\begin{cases} x \geq -2 \\ y < 4 \end{cases}$$

$x \geq -2$
a solid vertical line that passes through $x = -2$

shaded to the right of $x = -2$

$y < 4$
a broken horizontal line that passes through $y = 4$

shaded down from $y = 4$



The square region is the intersection region as a result, it is the solution.

Ex:

Solve the system of linear inequalities

$$\begin{cases} 3x - 2y > 6 \\ y < -\frac{1}{2}x + 3 \end{cases}$$

$$3x - 2y > 6$$

1st step:

'>' broken line

2nd step:

$$3x - 2y \geq 6$$

$$x = 0$$

$$y = 0$$

$$3(0) - 2y = 6$$

$$3x - 2(0) = 6$$

$$-2y = 6$$

$$\frac{-2y}{-2} = \frac{6}{-2}$$

$$y = -3$$

$$3x = 6$$

$$\frac{3x}{3} = \frac{6}{3}$$

$$x = 2$$

3rd step:

Test point (0,0)

$$3x - 2y > 6$$

$$3(0) - 2(0) > 6$$

$$0 > 6 \text{ False}$$

Shade the region that doesn't include (0,0)

$$y < -\frac{1}{2}x + 3$$

1st step:

'<' broken line

2nd step:

$$y = \frac{1}{2}x + 3$$

$$x = 0$$

$$x = 6$$

$$y = \frac{1}{2}(0) + 3$$

$$y = -\frac{1}{2}(6) + 3$$

$$y = 3$$

$$= -3 + 3$$

$$y = 0$$

points (0,3) and (6,0)

3rd step:

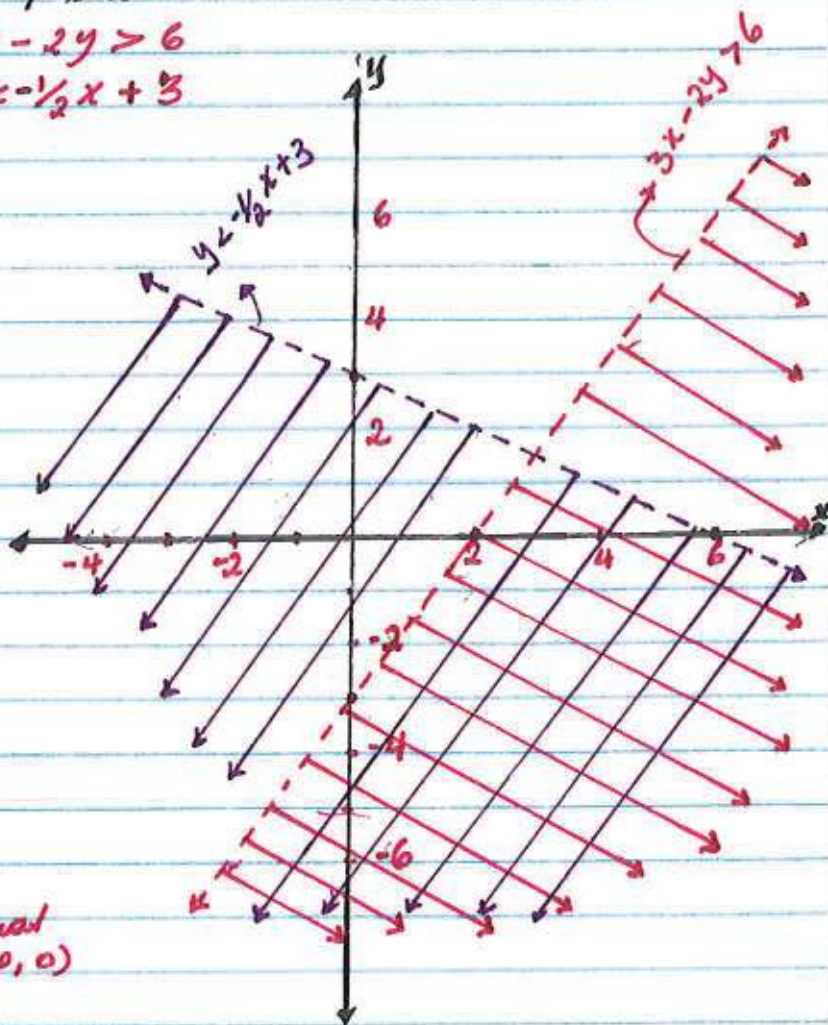
Test point (0,0)

$$y < -\frac{1}{2}(x) + 3$$

$$0 < -\frac{1}{2}(0) + 3$$

$$0 < 3 \text{ TRUE}$$

shade towards (0,0)



Exponential Function

Functions are formulas that relates two or more things.

Some functions are a bit more complex than others.

Exponential functions are a type of functions that correlates "two" variables in a special way.

Before we get to exponential functions, it is important to go over powers.

$$y = a(b)^x$$

Diagram illustrating the components of the exponential function $y = a(b)^x$:

- Exponent:** x
- Base:** b
- Coefficient:** a

Exponent: The exponent tells us how many times we need to multiply the base by itself.

Base: The base gets to be multiplied by itself as many times as the exponent.

Coefficient: The coefficient multiplies the final outcome of the base and the exponent.

Examples:

a) 4^3

$$= \underbrace{4 * 4 * 4}_3$$

$$= 64$$

b) $2(3)^4$

$$= 2(3 * 3 * 3 * 3)$$
$$= 2(81)$$
$$= 162$$

c) $-3(2)^5$

$$= -3(2 * 2 * 2 * 2 * 2)$$
$$= -3(32)$$
$$= -96$$

Fundamental Properties of exponents:

These are 5 fundamental properties of exponents.

1. $x^0 = 1$

Anything the power of '0' is 1.
Exception would be $0^0 \neq 1$

2. $x^{-n} = \frac{1}{x^n}$

We can always convert a negative exponent into positive by making it a reciprocal.

3. $x^a * y^a = (xy)^a$
OR
 $x^a / y^a = (x/y)^a$

If two powers have the same exponents and they are connected by either multiplication or division, we can combine them under one exponent accordingly.

4. $x^a * x^b = x^{(a+b)}$
 $x^a / x^b = x^{(a-b)}$

If two powers have the same base and connected by multiplication, we can combine them by adding the exponents and if they are connected by division we can subtract their exponents.

5. $(x^m)^n = x^{(mn)}$

If we have two exponents for one base, then we can combine the two exponents by multiplication.

Fundamental properties of exponents summary:

1.	$x^0 = 1$
2.	$x^{-n} = \frac{1}{x^n}$
3.	$x^a \times y^a = (xy)^a$ $x^a / y^a = (x/y)^a$
4.	$x^a \times x^b = x^{(a+b)}$ $x^a / x^b = x^{(a-b)}$
5.	$(x^m)^n = x^{(mn)}$

Examples:

a) simplify the following,

$$\begin{aligned} -3x^{-2} &= -3 \left(\frac{1}{x^2} \right) \\ &= \boxed{\frac{-3}{x^2}} \end{aligned}$$

b) $4x^4y^2x$

$$\begin{aligned} &= 4(x^4 \times x)y^2 \\ &= 4(x^{4+1})y^2 \\ &= \boxed{4x^5y^2} \end{aligned}$$

c) $\frac{2x^3y^{-2}}{x^2y}$

$$\begin{aligned} &= 2 \left(\frac{x^3}{x^2} \right) \left(\frac{y^{-2}}{y^1} \right) \\ &= 2(x^{3-2})(y^{-2-1}) \\ &= 2x^1y^{-3} \\ &= 2x \left(\frac{1}{y^3} \right) \\ &= \boxed{\frac{2x}{y^3}} \end{aligned}$$

d) $\frac{3x^3y^3z^3}{x^2z}$

$$\begin{aligned} &= 3 \left(\frac{x^3}{x^2} \right) \left(\frac{z^3}{z} \right) \left(\frac{1}{y^3} \right) \\ &= 3x^{3-(2)}z^{(3-1)} \cdot \frac{1}{y^3} \\ &= 3x^5y^{\frac{1}{3}}z^2 \\ &= \boxed{\frac{3x^5z^2}{y^3}} \end{aligned}$$

e) $(3x^2y)^2 + \frac{(y^2)^2}{3}$

$$\begin{aligned} &= (3)^2(x^2)^2(y^2)^2 + \frac{(y^2)^2}{3} \\ &= 9x^4y^2 \cdot y^4 + \frac{y^4}{3} \\ &= \left(\frac{9}{3} \right) x^4(y^{2+4}) \\ &= \boxed{3x^4y^6} \end{aligned}$$

More examples:
Simplify the following

Eg 1:

$$\begin{aligned} & (-2x^{-2})^3 x^6 \\ & (-2)^3 (x^{-2})^3 (x^6) \\ & -8 x^{-6} x^6 \\ & -8 x^{(-6+6)} \\ & -8 x^0 \\ & -8(1) \\ & \boxed{-8} \end{aligned}$$

Eg 2:

$$\begin{aligned} & \frac{4x^{-2} p^3 q^{-3} y^2}{(2x)^2 q^{-2}} \\ & \frac{4x^{-2} q^{-3} p^3 y^2}{4x^2 q^{-2}} \\ & \left(\frac{4}{4}\right) \left(\frac{x^{-2}}{x^2}\right) \left(\frac{q^{-3}}{q^{-2}}\right) (p^3 y^2) \\ & 1(x^{-2-2})(q^{-3-(-2)}) p^3 y^2 \\ & x^{-4} q^{-1} p^3 y^2 \\ & \left(\frac{1}{x^4}\right) \left(\frac{1}{q}\right) p^3 y^2 \\ & \boxed{\frac{p^3 y^2}{x^4 q}} \end{aligned}$$

Eg 3:

$$\begin{aligned} & \frac{(2x^2y)^3}{3x} * \frac{y^3}{x^2} \\ & \frac{(2)^3 (x^2)^3 (y)^3 (y)^3}{3x \cdot x^2} \\ & \frac{\frac{1}{3} (x^{-6}) y^{-3} y^3}{3x^3} \\ & \frac{1}{(8*3)} (x^{-6-3}) (y^{-3+3}) \\ & \frac{1}{24} x^{-9} y^0 \\ & \boxed{\frac{1}{24x^9}} \end{aligned}$$

Eg 4:

$$\frac{(-2x^{-2}y)^2 p^4}{x} \div \frac{(4xy^{-1})^{-2}}{p^{-3}}$$

1st we need to change the division into multiplication by replacing the divisor by its reciprocal.

$$\begin{aligned} & \frac{(-2x^{-2}y)^2 p^4}{x} * \frac{p^{-3}}{(4xy^{-1})^{-2}} \\ & \frac{(-2)^2 (x^{-2})^2 y^2 p^4}{x} * \frac{p^{-3}}{(4)^2 (x)^2 (y^{-1})^2} \end{aligned}$$

$$\begin{aligned} & \frac{4x^{-4} y^2 p^4}{x} * \frac{p^{-3}}{\frac{1}{16} \cdot \frac{1}{x^2} \cdot y^2} \\ & \frac{4 \left(\frac{1}{x^4}\right) y^2 p^4}{x} * \frac{\left(\frac{1}{p^3}\right)}{y^2 / 16x^2} \\ & \frac{4y^2 p^4}{x^4 \cdot x} * \frac{16x^2}{y^2 p^3} \end{aligned}$$

$$\begin{aligned} & (4*16) \left(\frac{y^2}{y^2}\right) \left(\frac{p^4}{p^3}\right) \left(\frac{x^2}{x^4 \cdot x^1}\right) \\ & (64) (y^{2-2}) (p^{4-3}) (x^{2-5}) \end{aligned}$$

$$\begin{aligned} & 64 y^0 p^1 x^{-3} \\ & 64 * 1 * p \left(\frac{1}{x^3}\right) \\ & \boxed{\frac{64p}{x^3}} \end{aligned}$$

Scientific Notation

Scientific notation is a special way of writing very small or large numbers.

A scientific notation uses a combination of numbers between 0 and 9 and powers with base 10.

$$= a \times 10^b$$

↓
needs to be between 1 and 9

Exponent.

Example:

a) $2,434,412$
 $= 2.434412 \times 1,000,000$
 $= 2.43412 \times 10^6$

b) $240,000$
 $= 2.4 \times 100,000$
 $= 2.4 \times 10^5$

c) 51
 $= 5.1 \times 10$
 $= 5.1 \times 10^1$

For example if we have a very small negative number, then, we follow the same principle except the exponent becomes a negative.

$$0.0023$$

we need to put the decimal between 2 and 3 by moving 3 units to the right.

$$0.0023 = 2.3 \times 10^{-3}$$

d) 0.00012
decimal needs to be between 1 and 2
→ move 4 unit to the right.

$$0.00012 = 1.2 \times 10^{-4}$$

e) 0.00000253
decimal needs to be between 2 and 5.

$$0.00000253 = 2.53 \times 10^{-6}$$

Multiplying numbers in
scientific notation:

Example:

$$\begin{aligned} \text{a) } (2.3 \times 10^5) \times (1.2 \times 10^2) \\ &= (2.3 \times 1.2) (10^5 \times 10^2) \\ &= (2.76) (10^{5+2}) \\ &= 2.76 \times 10^7 \end{aligned}$$

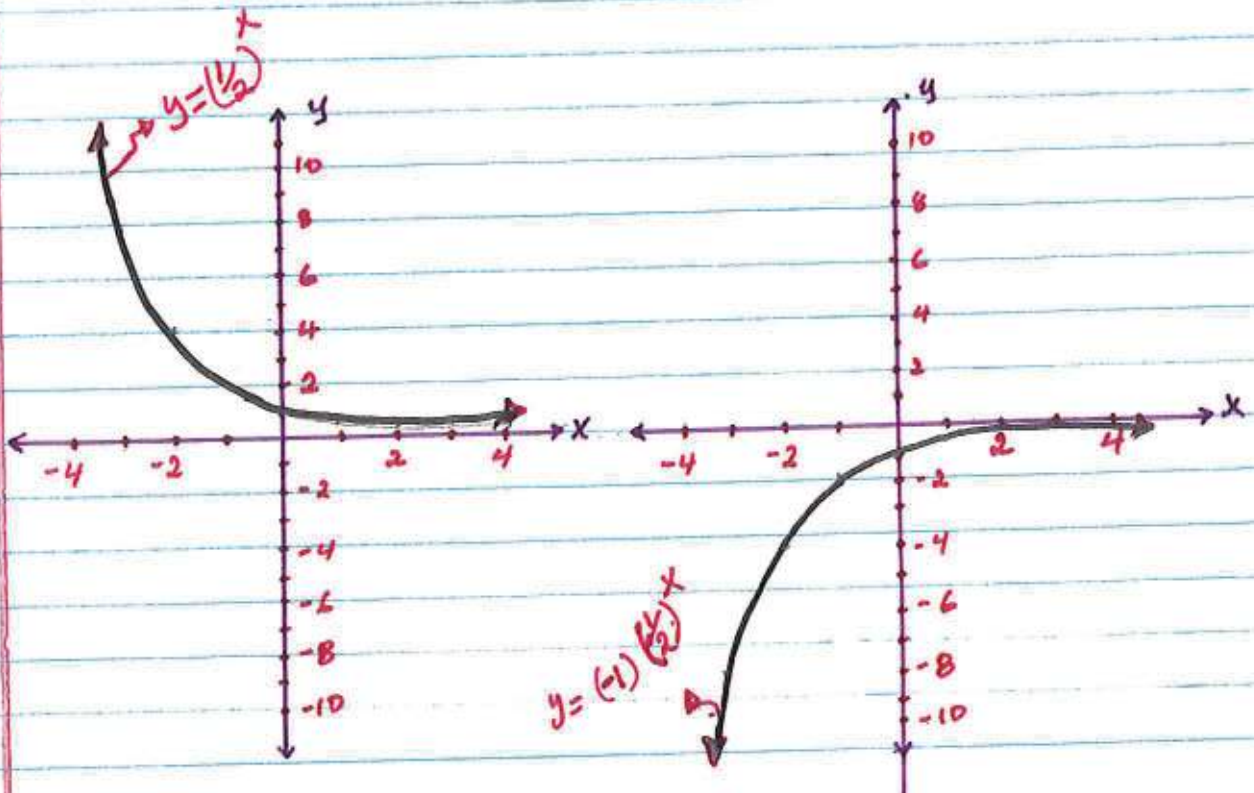
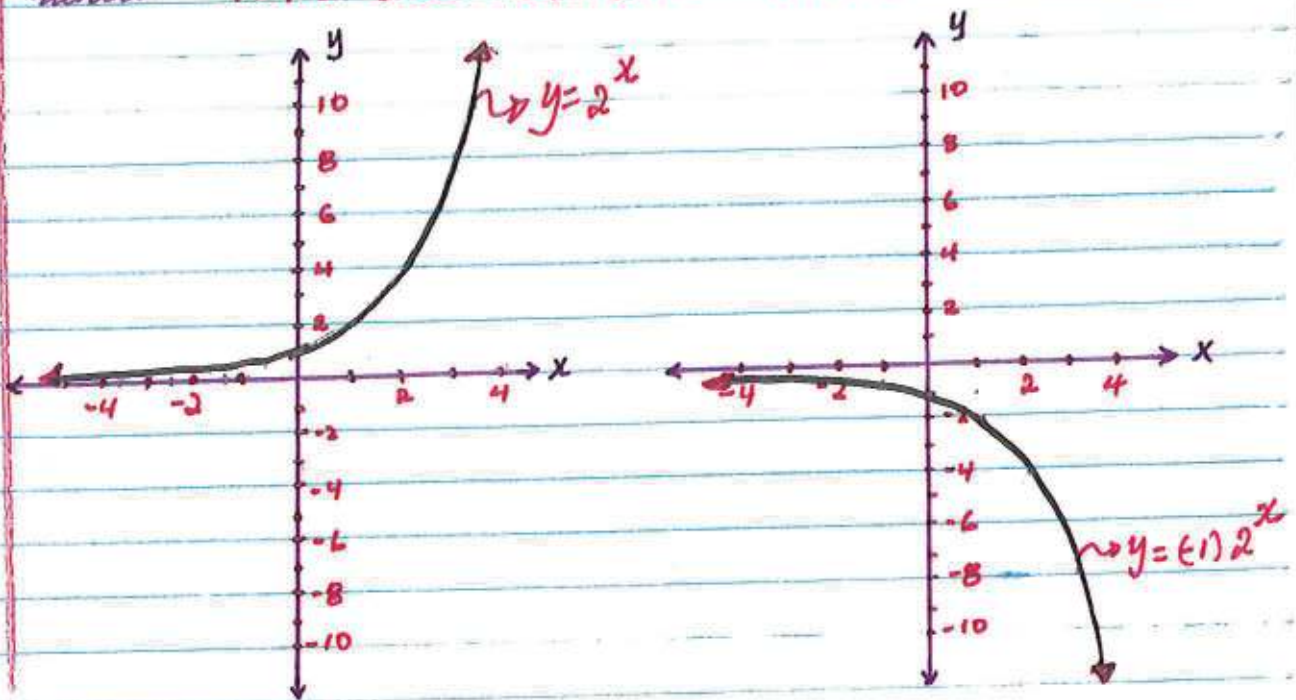
$$\begin{aligned} \text{b) } (4.2 \times 10^{-3}) \times (2.5 \times 10^3) \\ &= (4.2 \times 2.5) (10^{-3} \times 10^3) \\ &= 10.5 \times 10^{-3+3} \\ &= 10.5 \times 10^0 \\ &= 10.5 \times 10 \\ &= 105 \times 10^1 \end{aligned}$$

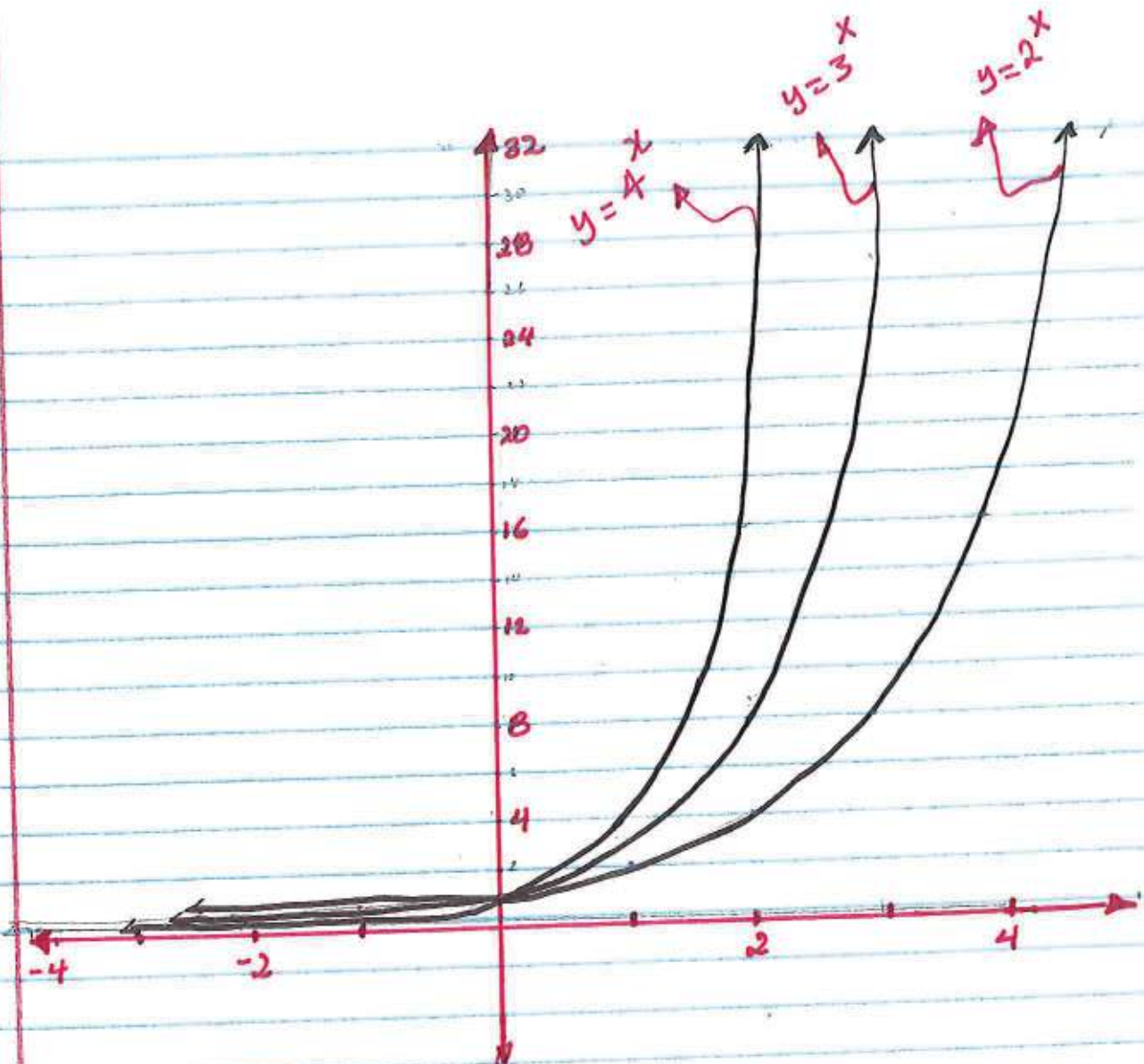
$$\begin{aligned} \text{c) } (1 \times 10^5) (2.3 \times 10^{-2}) \\ &(1 \times 2.3) (10^5 \times 10^{-2}) \\ &2.3 (10^{5+(-2)}) \\ &2.3 \times 10^3 \\ &2.3 \times 10^3 \end{aligned}$$

$$\begin{aligned} \text{d) } (-2 \times 10^3) / (4 \times 10^2) \\ &\left(\frac{-2}{4}\right) \left(\frac{10^3}{10^2}\right) \\ &(-0.5) (10^{3-(-2)}) \\ &(-0.5) (10^{3+2}) \\ &-0.5 \times 10^5 \\ &-5 \times 10^4 \end{aligned}$$

Exponential function and their graphs.

Exponential functions are the form $y = a(b)^x$ where $a \neq 0$ and $b \neq 1$ and $x \in \mathbb{R}$





Notice that the graph gets closer and closer as "b" $y = a(b)^x$ increases from $1 + 0 \rightarrow \infty$

When b is between 1 and ∞ , the exponential function is commonly known as **Exponential Growth**

When b is between 0 and 1, the exponential function is commonly known as **Exponential Decay**.

In this case 'b' is called a growth factor and a decay factor respectively.

How to find exponential functions given two points:

Ex 1: Find the function (exponential) that passes through $(1, 6)$ and $(3, 24)$

Let $f(x) = a(b)^x$ be the function.
Now, we just need to find the values of a and b and we are done.

$$y = a \cdot (b)^x$$

from the two points, we get two equations

$$(1, 6)$$

$$y = a(b)^x$$

$$6 = a(b)^1$$

$$6 = ab$$

$$(3, 24)$$

$$y = a(b)^x$$

$$24 = a(b)^3$$

$$24 = ab^3$$

now, we can write the first equation in terms of a , i.e.

$$\frac{6}{b} = \frac{ab}{b}$$

$$a = \frac{6}{b}$$

--- substitute this in the 2nd Eq.

$$24 = ab^3$$

$$24 = \left(\frac{6}{b}\right)b^3$$

$$24 = 6\left(\frac{b^3}{b}\right)$$

$$24 = 6b^{3-1}$$

$$\frac{24}{6} = \frac{6b^2}{6}$$

$$b^2 = 4$$

$$b = \pm\sqrt{4}$$

$$b = \pm 2$$

$$a = \frac{6}{b}$$

$$a = \pm\frac{6}{2}$$

$$a = \pm 3$$

$$\boxed{\begin{aligned} f(x) &= 3(2)^x \\ \text{OR} \\ f(x) &= -3(2)^x \end{aligned}}$$

Working with Polynomials

A polynomial function is given by the form

$$P(x) = A_n x^n + A_{n-1} x^{n-1} + A_{n-2} x^{n-2} + \dots + A_0$$

It is very important to understand some of the vocabularies and terminologies that are associated with a polynomial function.

$$P(x) = A_n x^n + A_{n-1} x^{n-1} + \dots + A_0$$

Annotations for the polynomial equation above:

- degree: points to the exponent n in x^n .
- coefficient: points to A_n .
- variable: points to x .
- term: points to the entire expression $A_n x^n$.
- constant: points to A_0 .

$$A_n, A_{n-1}, A_{n-2}, \dots, A_0$$

coefficients of the polynomial

$$n, n-1, n-2, \dots, 0$$

degrees of the polynomial

$$A_n x^n, A_{n-1} x^{n-1}, A_{n-2} x^{n-2}, \dots$$

terms of the polynomial

$$A_0$$

constant, it is also a term.

Examples of polynomials

a) $3x^4 + 2x^5 - 3$

b) $3x^2 - 1$

c) $-\frac{1}{3}x^2 + 2x$

d) $x^5 - 1x^2 + 6$

e) 4

This are some examples of polynomials.

Standard form of a polynomial:

when we write a polynomial, with the term with the highest degree, comes 1st and the rest of the polynomial terms follow in descending order:

For example

$$p(x) = 3x^4 + 2x^5 - 3 + 4x$$

Not standard form

↓

$$p(x) = 2x^5 + 3x^4 + 4x - 3$$

standard form.

More examples on terminologies.

a) $p(x) = 2x^3 - 4x + 1$

↳ has three terms

↳ the degree of the polynomial is 3.

b) $p(x) = -4x^4 + 2$

↳ has two terms

↳ the degree of the polynomial is 4.

c) $p(x) = \frac{1}{2}x + 1$

↳ has two terms

↳ the degree of the polynomial is 1.

Naming Polynomials

We can name polynomials based on the number of terms the polynomial has and the degree of the polynomial.

* Remember the degree of a polynomial is the highest exponent.

Name Based on # of Terms

1 term	Monomial
2 terms	Binomial
3 terms	Trinomial
4 terms	Polynomial
4 more	

Name Based on Degree

0 degree	Constant
1 st degree	Linear
2 nd degree	Quadratic
3 rd degree	Cubic
4 th degree	Quartic
5 th degree	Quintic
6 th degree	6 th degree polynomial.
4 more	

Example:

Name the following polynomials

- | | |
|------------------------------|-------------------------------------|
| a) $4x^3 - 2x$ | a cubic binomial |
| b) $3x^2 + 2x - 1$ | a quadratic trinomial |
| c) x^4 | a quartic monomial |
| d) $\frac{1}{2}x^5 + 2x - 1$ | a quintic trinomial |
| e) $x^7 + 4$ | a 7 th degree binomial |
| f) -2 | a constant monomial |
| g) $x^6 - 2x^3 + 4x^2 - 2$ | a 6 th degree polynomial |

Simplifying Polynomials

Definition:

Like terms:

like terms are terms with the same variables and exponents / degree.

for example,

- $2x^2$ & $-3x^2$
- $4p^3$ & p^3
- $-2q^5$ & $10q^5$
- 7 & 2 etc.

We can always combine like terms based on how they are connected. For example if two like terms are combined by addition, then we can add their respective coefficients i.e. $2x^3 + 5x^3 \rightarrow 7x^3$

Simplifying polynomials is basically combining like terms, if there are any like terms.

Eg 1: $p(x) = 3x^3 - 2x^2 + 4x^2 - x + 2$

$$p(x) = 3x^3 + 2x^2 - x + 2$$

these two like terms.

$$-2x^2 \text{ and } 4x^2$$

we can combine -2 & 4 and take up one of the x^2 's.

$$(-2 + 4)x^2$$

$$2x^2$$

Eg 2: $p(x) = 2 + x^2 - x^3 + 2x^2$

$$p(x) = 2 + 3x^2 - x^3$$

two like terms; x^2 & $2x^2$

rearranging the terms to standard form

$$(1 + 2)x^2$$

$$3x^2$$

$$p(x) = -x^3 + 3x^2 + 2$$

Adding and Subtracting Polynomials.

We can add polynomials by adding like terms:

Eg1. Let $f(x) = 2x^3 + x$
 $g(x) = -3x^2 + 2x + 1$

$$f(x) + g(x) = (2x^3 + x) + (-3x^2 + 2x + 1)$$

1st lets make sure we get rid off the parentheses. since there are no numbers multiplying each polynomial, we can just eliminate the parentheses.

$$= 2x^3 + x + (-3x^2) + 2x + 1$$

$$= 2x^3 + x - 3x^2 + 2x + 1$$

$$= \boxed{2x^3 - 3x^2 + 3x + 1}$$

Eg2: Let $f(x) = 10x^4 + 2x^3 - 3$

$$h(x) = -10x^4 - 2x^3 + 5$$

$$f(x) + h(x) = (10x^4 + 2x^3 - 3) + (-10x^4 - 2x^3 + 5)$$

$$= 10x^4 + 2x^3 - 3 - 10x^4 - 2x^3 + 5$$

like terms

$$= (10x^4 - 10x^4) + (2x^3 - 2x^3) + (-3 + 5)$$

$$= 0 + 0 + 2$$

$$= \boxed{2}$$

Eg3: Let $f(x) = x^5$

$$g(x) = 2x^3 - x + 1$$

$$f(x) + g(x) = (x^5) + (2x^3 - x + 1)$$

$$= \boxed{x^5 + 2x^3 - x + 1}$$

Subtracting polynomial

let $f(x) = x^2 + 2$

$g(x) = -x^3 + 2x^2 - 5$

$$f(x) - g(x) = (x^2 + 2) - (-x^3 + 2x^2 - 5)$$

↳ notice that, the minus sign needs to be distributed over $g(x)$ before we simplify.

$$= (x^2 + 2) - (-x^3 + 2x^2 - 5)$$

$$= x^2 + 2 - (-x^3) - (2x^2) - (-5)$$

$$= x^2 + 2 + x^3 - 2x^2 + 5$$

$$= -x^2 + 7 + x^3$$

↳ Rearranging standard form.

$$= \boxed{x^3 - x^2 + 7}$$

let $f(x) = -x^5 + 10x^2 + 3x$

$g(x) = 2x^2 + 3x$

This time lets subtract $f(x)$ from $g(x)$

$$g(x) - f(x) = (2x^2 + 3x) - (-x^5 + 10x^2 + 3x)$$

$$= (2x^2 + 3x) - (-x^5 + 10x^2 + 3x)$$

$$= 2x^2 + 3x - (-x^5) - (10x^2) - (3x)$$

$$= 2x^2 + 3x + x^5 - 10x^2 - 3x$$

$$= (2x^2 - 10x^2) + (3x - 3x) + x^5$$

$$= -8x^2 + 0 + x^5$$

↳ Rearranging to standard form

$$= \boxed{x^5 - 8x^2}$$

Multiplying Polynomials

Multiplying polynomials is a multiple step process.

Let's start by multiplying a monomial with polynomials:

Eg: $4x^2 (2x^3 + 3x^2 - 1)$

The monomial ($4x^2$) is going to multiply every term in the trinomial ($2x^3 + 3x^2 - 1$)

Distributing $4x^2$

$$4x^2 (2x^3 + 3x^2 - 1)$$

Recall some of the properties of exponents with the same base: $x^3 \cdot x^4 = x^{3+4} = x^7$

$$(4x^2 \cdot 2x^3) + (4x^2 \cdot 3x^2) - (4x^2 \cdot 1)$$

$$(4 \cdot 2)(x^2 \cdot x^3) + (4 \cdot 3)(x^2 \cdot x^2) - (4 \cdot 1)x^2$$

$$8x^{2+3} + 12x^{2+2} - 4x^2$$

$$\boxed{8x^5 + 12x^4 - 4x^2}$$

Eg 2: $-3x^4 (-2x^2 + x^4 - 1)$

$$-3x^4 (-2x^2 + x^4 - 1)$$

$$(-3x^4 \cdot -2x^2) + (-3x^4 \cdot x^4) - (-3x^4 \cdot 1)$$

$$(-3 \cdot -2)(x^4 \cdot x^2) + (-3 \cdot 1)(x^4 \cdot x^4) - (-3 \cdot 1)x^4$$

$$6x^{4+2} + (-3)x^{4+4} - (-3)x^4$$

$$6x^6 - 3x^8 + 3x^4$$

rearranging to standard form

$$\boxed{-3x^8 + 6x^6 + 3x^4}$$

Multiplying Binomials with polynomials

Recall that a binomial has two terms.
Each term needs to be distributed over
the other polynomial. i.e.;

Eg: $(2x^2 - x)(x^3 + 2x^2 - 4)$

$$(2x^2 - x)(x^3 + 2x^2 - 4)$$

$$(2x^2)(x^3 + 2x^2 - 4) - x(x^3 + 2x^2 - 4)$$

$$(2x^2 * x^3) + (2x^2 * 2x^2) - (2x^2 * 4) + (-x * x^3) + (-x * 2x^2) - (-x * 4)$$

$$2x^{2+3} + 4x^{2+2} - 8x^2 - (x^{1+3}) - (2x^{1+2}) + 4x$$

$$2x^5 + 4x^4 - 8x^2 - x^4 - 2x^3 + 4x$$

like terms

$$2x^5 + 3x^4 - 2x^3 - 8x^2 + 4x$$

Eg: $(-x^5 + 3x^2)(x^2 - 5x)$

$$(-x^5 + 3x^2)(x^2 - 5x)$$

$$(-x^5 * x^2) - (x^5 * -5x) + (3x^2 * x^2) + (3x^2 * -5x)$$

$$(-x^{5+2}) - (-5x^{5+1}) + (3x^{2+2}) + (-15x^{2+1})$$

$$-x^7 - 5x^6 + 3x^4 - 15x^3$$

Remark:

Always remember the signs of each
term when you multiply
polynomials.

Factoring Polynomials

Factoring polynomials is the reverse process of multiplying polynomials.

Before we go any further with factoring polynomials, let's talk about the greatest common factor (GCF) since it's going to play an important role.

Prime factorization:

factors are numbers that multiply together to give a bigger number.

when factors are prime numbers, then we call the method prime factorization.

Eg: 4 and 3 are factors of 12

but they are not the prime factors of 12.

This is because 4 is not a prime number. If we change 4 into 2 and 2,

then 2, 2, and 3 are the prime factors of 12.

$$12 = 2 \times 2 \times 3.$$

$$\text{Eg: } 48 = 12 \times 4$$

$$= \underline{2 \times 2 \times 3} \times \underline{2 \times 2}$$

$$= 2 \times 2 \times 2 \times 2 \times 3$$

prime factors of 48.

Factor tree is one of the best ways to find the prime factors of any number.

How: Divide the number starting with 2, 3, ... unit, we end up with 1.

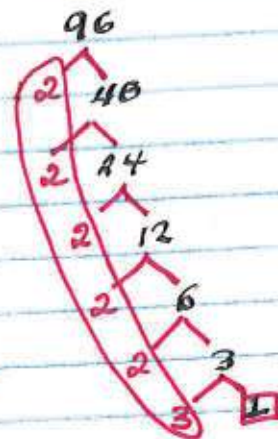
Eg: Factor Tree:



These are the prime factors of
 $36 = 2 \times 2 \times 3 \times 3$



$$54 = 2 \times 3 \times 3 \times 3$$



$$96 = 2 \times 2 \times 2 \times 2 \times 2 \times 3$$

Now that we have ~~seen~~ seen how prime factorization works, let's start factoring polynomials:

Eg: $15x^4 + 36x^2$

$$15x^4 = \underline{3} \times 5 \times \underline{x} \times \underline{x} \times \underline{x} \times \underline{x}$$

$$36x^2 = 2 \times 2 \times \underline{3} \times 3 \times \underline{x} \times \underline{x}$$

} Let's look for something that both polynomials have

They both have one 3 and two x's. and we can factor them out.

i.e.

$$15x^4 + 36x^2$$

$$\underline{3} \times 5 \times \underline{x} \times \underline{x} \times \underline{x} \times \underline{x} + 2 \times 2 \times \underline{3} \times 3 \times \underline{x} \times \underline{x}$$

whenever you factor out GCF, you need to leave the remaining terms inside a parenthesis. i.e.

$$3 \times x \times x [5 \times x \times x + 2 \times 2 \times 3]$$

$$3x^2 [5x^2 + 12]$$

If we distribute the $3x^2$ over $5x^2 + 12$, we will get our original terms.

Eg:

$$54x^3 - 18x^2 + 36x$$

$$54x^3 = \underline{2} * \underline{3} * \underline{3} * \underline{3} * \underline{x} * \underline{x} * \underline{x}$$

$$18x^2 = \underline{2} * \underline{3} * \underline{3} * \underline{x} * \underline{x}$$

$$36x = \underline{2} * \underline{3} * \underline{2} * \underline{3} * \underline{x}$$

$$\text{GCF} = 2 * 3 * x = 6x$$

$$54x^3 - 18x^2 + 36x = 6x [3 * 3 * x * x - 3 * x + 2 * 3]$$
$$= 6x [9x^2 - 3x + 6]$$

Eg: $15x^{10} - 25x^4 + 45x^2$

$$15x^{10} = 3 * \underline{5} * \underline{x} * \underline{x} * \underline{x} * \underline{x} * \underline{x} * \underline{x} * \underline{x} * \underline{x} * \underline{x} * \underline{x}$$

$$25x^4 = 5 * \underline{5} * \underline{x} * \underline{x} * \underline{x} * \underline{x}$$

$$45x^2 = 3 * \underline{3} * \underline{5} * \underline{x} * \underline{x}$$

$$\text{GCF} = 5 * x * x$$

$$= 5x^2$$

Factoring $5x^2$ from $15x^{10} - 25x^4 + 45x^2$ yields

$$5x^2 [3x^8 - 5x^2 + 9]$$

If we distribute $5x^2$ over $(3x^8 - 5x^2 + 9)$ we get.

$$5x^2 (3x^8 - 5x^2 + 9)$$
$$(5x^2 * 3x^8) - (5x^2 * 5x^2) + (5x^2 * 9)$$
$$(5 * 3 x^{2+8}) - (5 * 5 x^{2+2}) + (5 * 9 x^2)$$
$$15x^{10} - 25x^4 + 45x^2$$

which is the original polynomial.

That is why, factoring is the reverse process of multiplying polynomials.

Special Cases

$$1. (x+y)^2 = x^2 + 2xy + y^2$$

$$2. (x-y)^2 = x^2 - 2xy + y^2$$

$$3. x^2 - y^2 = (x-y)(x+y)$$

$$4. x^3 + y^3 = (x+y)(x^2 - xy + y^2)$$

$$5. x^3 - y^3 = (x-y)(x^2 + xy + y^2)$$

Examples:

$$\begin{aligned} \text{Eg 1. } (x+2)^2 &= x^2 + 2(x)(2) + 2^2 \\ &= x^2 + 4x + 4 \end{aligned}$$

$$\begin{aligned} \text{Eg 2. } (p-3)^2 &= \\ &= p^2 - 2(p)(3) + 3^2 \\ &= p^2 - 6p + 9 \end{aligned}$$

$$\begin{aligned} \text{Eg 3. } x^2 - 25 &= \\ &= x^2 - 5^2 \\ &= (x-5)(x+5) \end{aligned}$$

$$\begin{aligned} \text{Eg 4. } x^3 + 8 &= \\ &= x^3 + 2^3 \\ &= (x+2)(x^2 + x(2) + 2^2) \\ &= (x+2)(x^2 + 2x + 4) \end{aligned}$$

$$\begin{aligned} \text{Eg 5. } x^3 - 64 &= \\ &= x^3 - 4^3 \\ &= (x-4)(x^2 + x(4) + 4^2) \\ &= (x-4)(x^2 + 4x + 16) \end{aligned}$$

These examples demonstrate how powerful these special cases can be, when we need to expand or factor polynomials that satisfies the above conditions.

FACTORIZING TRINOMIALS

$P(x) = ax^2 + bx + c$, with degree 2.
where $a, b,$ and c
are integer coefficients

In order to understand factoring polynomials of the form $ax^2 + bx + c$, 1st let's see an example that demonstrates a trinomial of that form and its factors.

$(x+2)$ and $(x+3)$ are factors of $x^2 + 5x + 6$
if we multiply $(x+2)$ by $(x+3)$, we get

$$\begin{array}{r} x \times x + x \times 3 + 2 \times x + 2 \times 3 \\ x^2 + 3x + 2x + 6 \\ \hline x^2 + 5x + 6 \end{array}$$

As a result $(x+2)$ & $(x+3)$
are factors of $x^2 + 5x + 6$

Now:

The question is how could we find these factors given the trinomial $x^2 + 5x + 6$

Method called sum & product.
trinomial form $ax^2 + bx + c$

we look for two numbers when we add them we get "b" and when we multiply them, we get "a*c"

$$ax^2 + bx + c$$

$$\begin{array}{cc} \text{sum} & \text{product} \\ b & a \cdot c \end{array}$$

Eg: $x^2 + 5x + 6$

sum
5

product
 $1 \times 6 = 6$

The numbers that we are looking for are 3 and 2 since

it's convenient & we look for the factors of "6"

sum
 $2+3=5$

product
 $2 \times 3 = 6$

$7 = 1+6$

$1 \times 6 = 6$

$5 = 2+3$

$2 \times 3 = 6$

$-7 = -1+6$

$-1 \times 6 = 6$

$-5 = -2+3$

$-2 \times 3 = 6$

Now we can replace the 5x in $x^2 + 5x + 6$ by $2x + 3x$

$x^2 + 5x + 6$

$x^2 + 2x + 3x + 6$

now we can group $x^2 + 2x$ and $3x + 6$

$x^2 + 2x$ + $3x + 6$

- $x^2 + 2x$ has an 'x' as factor

$x^2 + 2x = x(x+2)$

- $3x + 6$ has 3 as a factor

$x(x+2)$ + $3(x+2)$

$(x+2)$ is a factor

for both $x(x+2)$ and $3(x+2)$

$x(x+2)$ + $3(x+2)$

$(x+2)(x+3)$

summary

$x^2 + 5x + 6$

sum product

$x^2 + 2x + 3x + 6$

5 6

$x(x+2)$ + $3(x+2)$

the #'s 2 & 3

$(x+2)(x+3)$

$(x^2 + 5x + 6) = (x+2)(x+3)$

Eg

$$x^2 - 4x + 3$$

sum
-4

product

$$1 \times 3 = 3$$

#'s

-1 and -3

$$\begin{aligned} & \underline{x^2 - 1x - 3x + 3} \\ & x(x-1) - 3(x-1) \\ & (x-1)(x-3) \end{aligned}$$

factors: $(x-1)(x-3)$

Eg

$$x^2 + 8x - 20$$

sum
8

product

$$1 \times -20 = -20$$

#'s

10 and -2

$$\begin{aligned} & \underline{x^2 - 2x + 10x - 20} \\ & x(x-2) + 10(x-2) \\ & (x-2)(x+10) \end{aligned}$$

$$8x = -2x + 10x$$

Now let's try when the leading coefficient "a" is different from 1.

Eg.

$$2x^2 + 5x - 7$$

sum
5

product

$$2 \times -7 = -14$$

#'s

7 and -2

$$\begin{aligned} & \underline{2x^2 - 2x + 7x - 7} \\ & 2x(x-1) + 7(x-1) \\ & (x-1)(2x+7) \end{aligned}$$

Eg

$$-3x^2 - 12x + 15$$

sum
-12

product

$$-3 \times 15 = -45$$

#'s

-15 and 3

$$\begin{aligned} & -3x^2 - 15x + 3x + 15 \\ & -3x(x+5) + 3(x+5) \\ & (x+5)(-3x+3) \end{aligned}$$

Remark: Not every trinomial has factors of these form

Eg.

$$x^2 + 1x + 2$$

doesn't have factors.

Factoring polynomials by Grouping:

Sometimes we can factor polynomial of some form by grouping.

↳ grouping means, gathering terms that have some common factors.

Eg 1:

$$3x^4 + 4x^3 + 6x^2 + 8x$$

Here, we can group $3x^4 + 4x^3$ and $6x^2 + 8x$

$$\underline{3x^4 + 4x^3} + \underline{6x^2 + 8x}$$

$$\underline{x^3(3x+4)} + \underline{2x(3x+4)}$$

$$(3x+4)(x^3+2x)$$

Eg 2: $\underline{2x^5 + 3x^2} - \underline{2x^3 - 3}$
 $x^2(2x^3+3) - 1(2x^3+3)$
 $(2x^3+3)(x^2-1)$

Eg 3: $x^8 + 4x^4 - 5$

To factor the above polynomial
1st let's substitute $x^4 = z$

$$(x^4)^2 + 4(x^4) - 5$$

$$z^2 + 4z - 5$$

↳ now let's use
sum & product.

$$z^2 - 1z + 5z - 5$$

$$z(z-1) + 5(z-1)$$

$$(z-1)(z+5) \rightarrow \text{now replace } z \text{ by } x^4$$

$$(x^4-1)(x^4+5)$$

sum	product
4	-5
#'s	5, and -1

Working with Quadratic Equations and Functions.

Quadratic functions are a specific type of polynomial functions where the highest degree is 2.

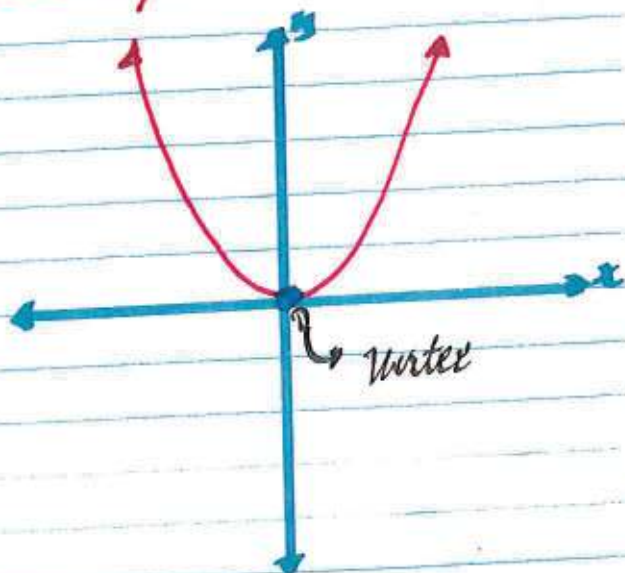
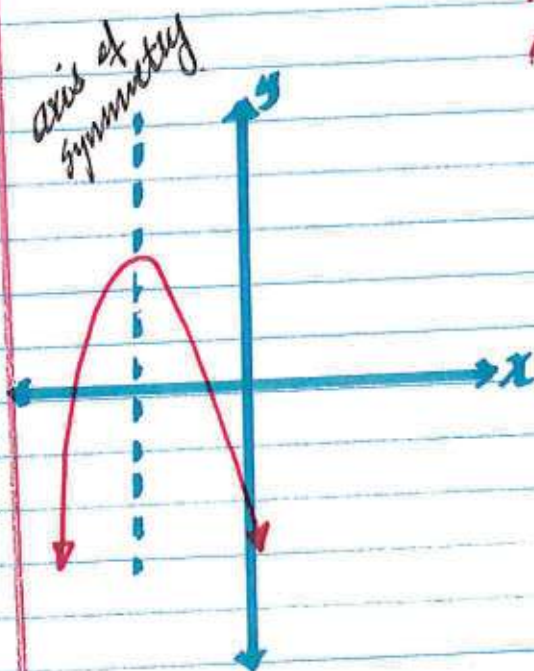
Let $Q(x)$ be a quadratic function. The standard form of a quadratic function is given by:

$$Q(x) = ax^2 + bx + c$$

where a, b, c are real coefficients and $a \neq 0$.

notice that, if $a=0$, we don't have an x^2 term, which results in a "no" quadratic function.

The graph of quadratic functions is called "parabola".



Axis of symmetry: Is a line that divides the parabola into two equal parts.

Vertex: Is a point on a given parabola where the graph turns direction. It is also known as turning point. It is normally given/represented by $V(h, k)$ where " h " is the x coordinate and " k " is the y -coordinate of turning point/vertex.

There are two forms of quadratic functions:

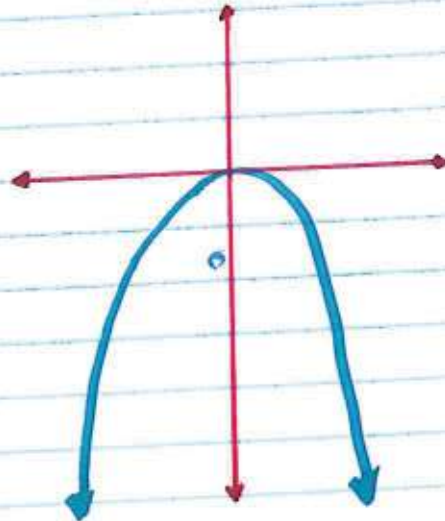
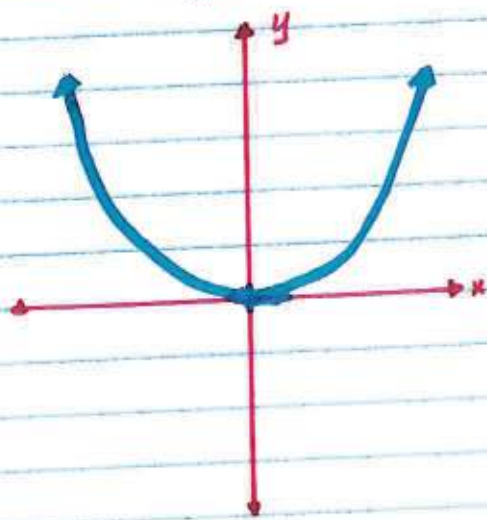
1. Standard form: $Q(x) = ax^2 + bx + c$

2. Vertex form: $Q(x) = a(x-h)^2 + k$

Two types of parabola:

upward parabola and
if $a > 0$
 \Rightarrow positive " a "

downward parabola
if $a < 0$
 \Rightarrow negative " a "



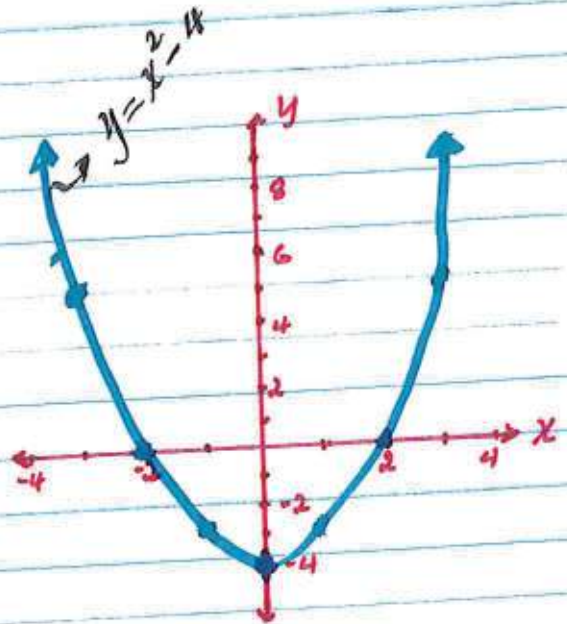
Drawing parabolas:

Eq: Draw the graph of

$$Q(x) = x^2 - 4$$

we can create a table

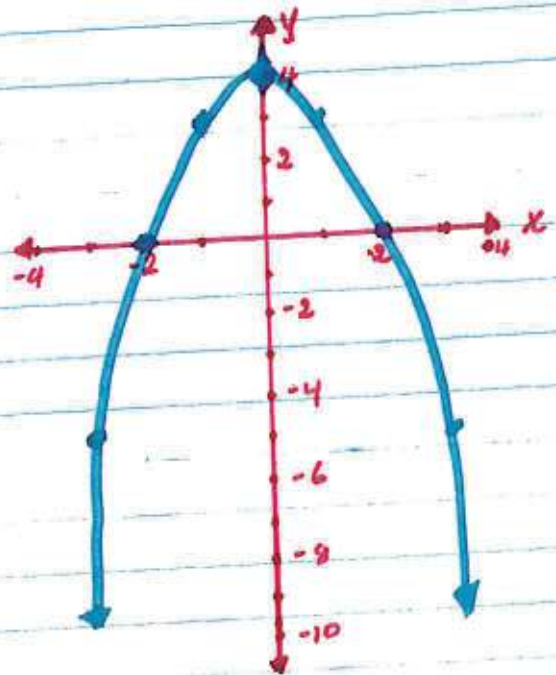
x	$Q(x)$	
-2	0	$Q(-2) = (-2)^2 - 4 = 0$
-1	-3	$Q(-1) = (-1)^2 - 4 = -3$
0	-4	$Q(0) = (0)^2 - 4 = -4$
1	-3	$Q(1) = (1)^2 - 4 = -3$
2	0	$Q(2) = (2)^2 - 4 = 0$
3	5	$Q(3) = (3)^2 - 4 = 5$



Eq: Draw the graph of

$$Q(x) = -x^2 + 4$$

x	$Q(x)$	
-3	-5	$Q(-3) = -(-3)^2 + 4 = -5$
-2	0	$Q(-2) = -(-2)^2 + 4 = 0$
-1	3	$Q(-1) = -(-1)^2 + 4 = 3$
0	4	$Q(0) = -(0)^2 + 4 = 4$
1	3	$Q(1) = -(1)^2 + 4 = 3$
2	0	$Q(2) = -(2)^2 + 4 = 0$
3	-5	$Q(3) = -(3)^2 + 4 = -5$



Drawing the graph of a quadratic equation/function is much more ~~easy~~ convenient if the function is given in vertex form. We just need to transform the parent function, which is $Q(x) = x^2$

Axis of Symmetry, vertex and vertex form of Quadratic functions.

Recall: Standard form; $a(x) = ax^2 + bx + c$; $a \neq 0$
Vertex form; $a(x) = a(x-h)^2 + k$

vertex (v); $V(h, k)$

$$h = \frac{-b}{2a} \quad k = \frac{4ac - b^2}{4a}$$

Axis of Symmetry:

$$x = \frac{-b}{2a}$$

Example:

Find the axis of symmetry and the vertex coordinates for $a(x) = x^2 + 8x + 12$.

1st let's write; $a, b,$ and c .

$$a = 1$$

$$b = 8$$

$$c = 12$$

Axis of Symmetry

$$x = \frac{-b}{2a} = \frac{-(8)}{2(1)} = \frac{-8}{2} = -4$$

$$\text{AOS; } x = -4$$

$$h = \frac{-b}{2a} = \frac{-(8)}{2(1)} = \frac{-8}{2} = -4$$

$$h = -4$$

$$k = \frac{4ac - b^2}{4a} = \frac{4(1)(12) - (8)^2}{4(1)}$$

$$= \frac{48 - 64}{4}$$

$$= \frac{-16}{4}$$

$$k = -4$$

Vertex form:

$$a(x) = a(x-h)^2 + k$$
$$= 1(x - (-4))^2 + (-4)$$

$$a(x) = 1(x+4)^2 - 4$$

Eq:

Find the vertex form of the following quadratic function

$$Q(x) = 2x^2 + 12x + 10$$

1st let's find a, b, and c

$$a = 2$$

$$b = 12$$

$$c = 10$$

$$h = \frac{-b}{2a} = \frac{-(12)}{2(2)} = \frac{-12}{4} = -3$$

$$h = -3$$

$$k = \frac{4ac - b^2}{4a} = \frac{4(2)(10) - (12)^2}{4(2)} = \frac{80 - 144}{8}$$

$$= \frac{-64}{8} = -8$$

$$k = -8$$

Vertex form =

$$a(x-h)^2 + k$$

$$Q(x) = 2(x - (-3))^2 + (-8)$$

$$Q(x) = 2(x+3)^2 - 8$$

If we expand the vertex form of any quadratic equation/function, we will get its standard form:

$$\text{Check: } = 2(x+3)^2 - 8$$

$$\text{1st expand } (x+3)^2$$

$$(x+3)(x+3)$$

$$= 2(x^2 + 6x + 9) - 8$$

$$= 2x^2 + 12x + 18 - 8$$

$$x^2 + 3x + 3x + 9$$

$$x^2 + 6x + 9$$

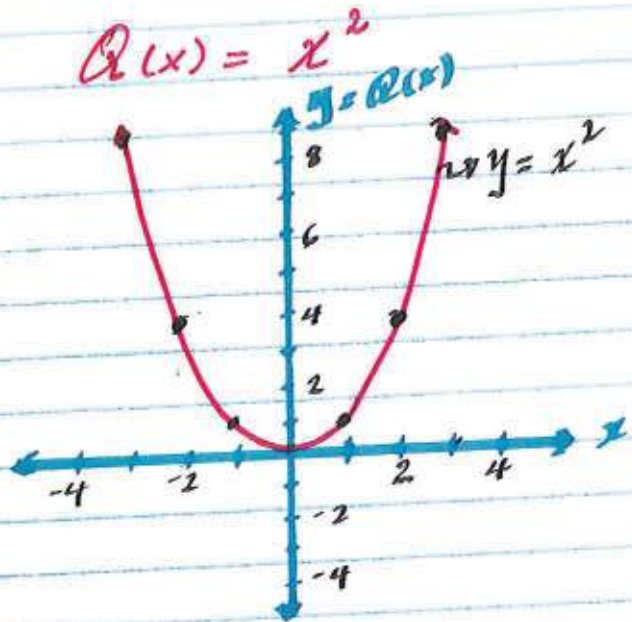
$$= 2x^2 + 12x + 10$$

which is what we started with.

Transforming a quadratic function graph.

The parent quadratic function is given by

x	Q(x)	
-3	9	$= (-3)^2 = 9$
-2	4	$= (-2)^2 = 4$
-1	1	$= (-1)^2 = 1$
0	0	$= (0)^2 = 0$
1	1	$= (1)^2 = 1$
2	4	$= (2)^2 = 4$
3	9	$= (3)^2 = 9$
4	16	$= (4)^2 = 16$



Recall the 1st example that we did.

$$Q(x) = x^2 + 8x + 12$$

we could draw $Q(x) = x^2 + 8x + 12$ in the same way by creating a table DR

we could transform its parent function (i.e. $Q(x) = x^2$).

In order to do that, we need its vertex form:

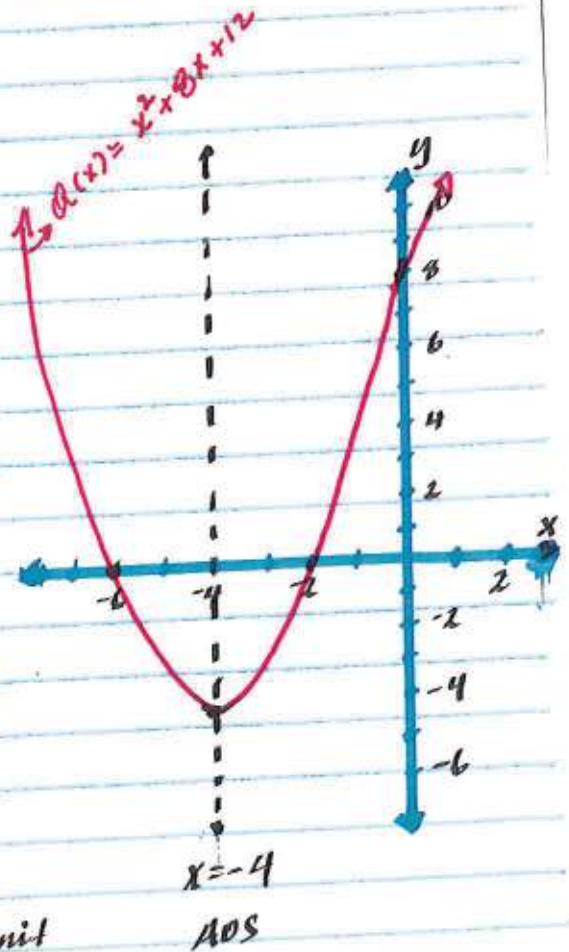
$$x^2 + 8x + 12 = 1(x+4)^2 - 4$$

In the vertex form

$$Q(x) = a(x-h)^2 + k$$

shrink or magnify left or right up or down

$$Q(x) = 1(x+4)^2 - 4 \rightarrow \begin{matrix} \text{The same} \\ \downarrow \\ \text{4 unit to the left} \end{matrix} \rightarrow \begin{matrix} \text{4 unit down} \end{matrix}$$



Eg:

Draw the graph of
 $Q(x) = -2x^2 + 8x + 24$
by translating its parent
function i.e. $(a(x) = x^2)$

1st we need to write $Q(x) = -2x^2 + 8x + 24$
in vertex form:

$$a = -2$$

$$b = 8$$

$$c = 24$$

$$h = \frac{-b}{2a} = \frac{-(8)}{2(-2)} = \frac{-8}{-4} = 2$$

$$h = 2$$

$$k = \frac{4ac - b^2}{4a} = \frac{4(-2)(24) - (8)^2}{4(-2)} = \frac{-192 - 64}{-8} = \frac{-256}{-8}$$

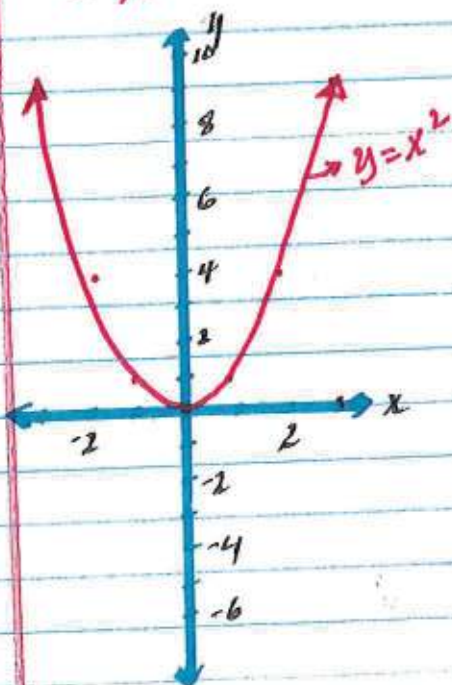
$$k = 32$$

$$Q(x) = -2(x-2)^2 + 32$$

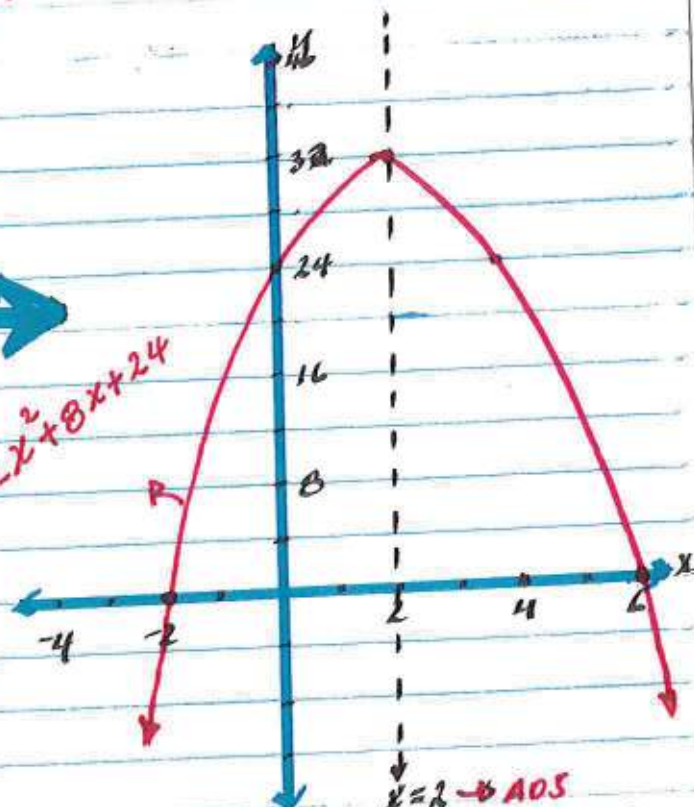
shifted 32 units up.

stretched
by 2 and
flipped

2 unit
to the
right



$$y = -2x^2 + 8x + 24$$



scaled by 4 to fit the graph

Solving Quadratic Equations

Recall: Quadratic functions are given by the form:

$$Q(x) = ax^2 + bx + c, a \neq 0$$

$$Q(x) = y$$

$$y = ax^2 + bx + c$$

When we set the value of y to 0, then the quadratic function becomes a quadratic equation.

$$ax^2 + bx + c = 0$$

↳ we can solve quadratic equation.

Remark:

If it was a quadratic function we can only graph it.

Solving quadratic equation is basically finding value/values of 'x' that makes the equation equals to 0.

Graphically, solutions of a quadratic equation are, points where the parabola crosses the x-axis, since the value of y equals to zero on the x-axis.

In the previous example

$$Q(x) = -2x^2 + 8x + 24$$

$$\Rightarrow -2x^2 + 8x + 24 = 0$$

$$\boxed{x = -2} \text{ and } \boxed{x = 6}$$

are the solutions of the quadratic equation $-2x^2 + 8x + 24$.

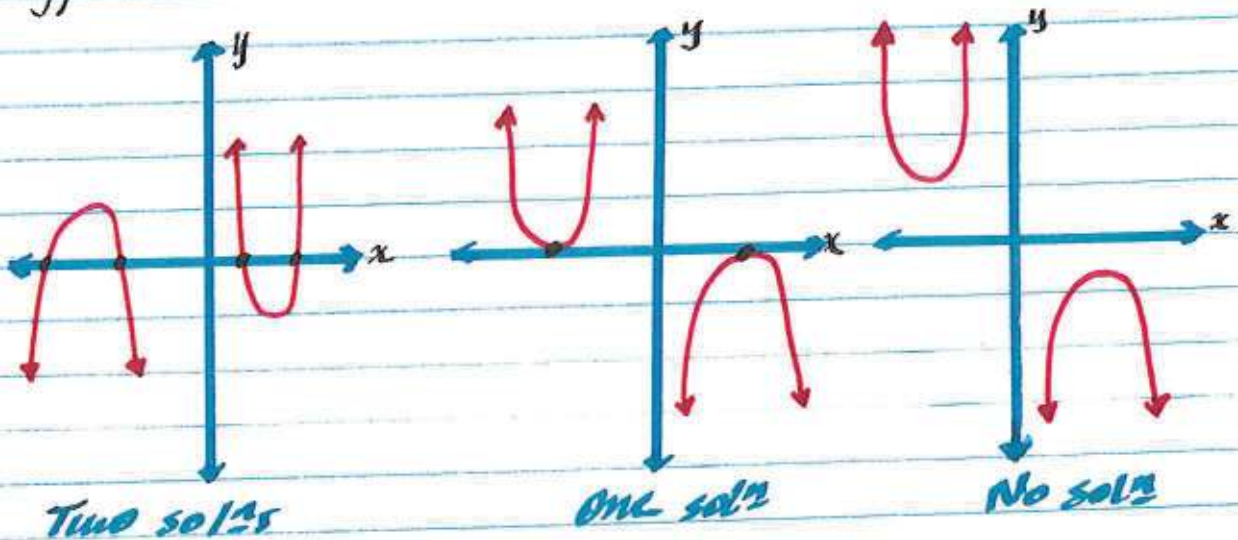
Type of Solutions

There are three different possible solutions for any given quadratic equation.

Type I: **two solutions**

Type II: **one solution**

Type III: **no solution**



Discriminant

The discriminant of a quadratic equation is the value of $b^2 - 4ac$.

This will help us to tell which type of solution it has.

$$D = b^2 - 4ac$$

If $D < 0$ (negative) **No solution**

$D = 0$ **One solution**

$D > 0$ (positive) **Two solutions**

The solution set of a quadratic equation is also called "Root".

Techniques solving quadratic equations

1. factoring
2. completing the square and
3. Quadratic formula

Factoring:

when we factor a quadratic equation, we get a product of two linear factors.
* when two factors multiply to give zero, one of them must equal to zero, at least. This helps to find the root/roots easily.

Eg 1:

$$\begin{aligned}x^2 + 5x + 6 &= 0 \\x^2 + 2x + 3x + 6 &= 0 \\x(x+2) + 3(x+2) &= 0 \\(x+2)(x+3) &= 0\end{aligned}$$

sum	product
5	6
#'s	2 and 3

now either $x+2=0$ OR $x+3=0$

$$-2 \quad -2$$

$$x = -2$$

$$-3 \quad -3$$

$$x = -3$$

$$S.S = \{-3, -2\}$$

Eg 2: $2x^2 - 6x + 4 = 0$

$$\begin{aligned}2x^2 - 2x - 4x + 4 &= 0 \\2x(x-1) - 4(x-1) &= 0 \\(x-1)(2x-4) &= 0\end{aligned}$$

sum	product
-6	$-2+4=8$
#'s	-2 and -4

either $(x-1)=0$ OR $2x-4=0$

$$+1 \quad +1$$

$$x = 1$$

$$+4 \quad +4$$

$$2x = 4$$

$$\frac{2x}{2} = \frac{4}{2}$$

$$S.S = \{1, 2\}$$

$$x = 2$$

Completing the Square.

Before we go over completing the square, let's talk about a perfect square.

A perfect square is a number or an expression where we can find another whole number or expression multiplied/squared by itself to get the original number.

Eg. 9 is a perfect square because $3 \times 3 = 9$

15 is not a perfect square since there is no whole number that can be multiplied by itself to give 15.

Eg. 49 is a perfect square because $7 \times 7 = 49$

Eg. $x^2 + 4x + 4$ is a perfect square, because $(x+2) \times (x+2) = x^2 + 4x + 4$.

Now we're ready to start completing the square.

Taking the square root of a perfect square.

$$\begin{aligned}x^2 &= 9 \\ \sqrt{x^2} &= \pm \sqrt{9} \\ x &= \pm 3\end{aligned}$$

To solve for x , we need to get rid of the square by taking the square root of both sides.

$$\begin{aligned}p^2 &= 64 \\ \sqrt{p^2} &= \pm \sqrt{64} \\ p &= \pm 8\end{aligned}$$

When we take square root make sure to put plus/minus ' \pm ' because it could either be positive or negative.

Solve $2x^2 + 8x - 10 = 0$ by completing the square.

$$\frac{-2x^2 + 8x - 10}{2} = \frac{0}{2}$$

1st divide everything by 2.

$$x^2 + 4x - 5 = 0$$

$$+5 \quad +5$$

2nd add the 5 to the other side

$$x^2 + 4x = 5$$

$$\frac{4}{2} = 2, \quad 2^2 = 4$$

Add 4 to both sides

Now we need to add a number to both sides of the equation. The number must make $x^2 + 4x$ a perfect square such number is found by dividing "b" in this case 4 by 2 and squaring the result

$$x^2 + 4x + 4 = 5 + 4$$

This is a perfect square and it can be written as

$$x^2 + 4x + 4 = (x+2)^2$$

$$(x+2)^2 = 9$$

$$\sqrt{(x+2)^2} = \pm \sqrt{9}$$

$$x+2 = \pm 3$$

$$-2 \quad -2$$

$$x = -2 \pm 3$$

take the square roots of both sides.

$$\text{So: } x = -2 + 3 \text{ and } x = -2 - 3$$

$$x = 1$$

$$x = -5$$

$$\text{S.S.} = \{-5, 1\}$$

Quadratic formula

We can use completing the square method to derive the quadratic formula.

$$ax^2 + bx + c = 0$$

$$\frac{ax^2}{a} + \frac{bx}{a} + \frac{c}{a} = \frac{0}{a}$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$x^2 + \frac{b}{a}x + \square = \frac{-c}{a} + \square$$

$$\frac{\frac{b}{a}}{2} = \frac{b}{2a} \Rightarrow \left(\frac{b}{2a}\right)^2$$

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{-c}{a} + \frac{b^2}{4a^2}$$

$$\Downarrow \quad \times \frac{4a}{4a}$$
$$\left(x + \frac{b}{2a}\right)^2 = \frac{-4ac}{4a^2} + \frac{b^2}{4a^2}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\sqrt{\left(x + \frac{b}{2a}\right)^2} = \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{\sqrt{4a^2}}; \quad \sqrt{4a^2} = 2a$$

$$x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$-\frac{b}{2a} \quad -\frac{b}{2a}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example 1:

Solve the quadratic equation $2x^2 + 2x - 24 = 0$ by using the quadratic formula.

1st let's find the coefficients a, b, c

$$a = 2$$

$$b = 2$$

$$c = -24$$

$$\text{Formula: } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

substitute the values of a, b, c

$$\begin{aligned} x &= \frac{-2 \pm \sqrt{(2)^2 - 4(2)(-24)}}{2(2)} \\ &= \frac{-2 \pm \sqrt{4 + 192}}{4} \\ &= \frac{-2 \pm \sqrt{196}}{4} = \frac{-2 \pm 14}{4} \end{aligned}$$

$$S.S = \{-4, 3\}$$

$$\begin{aligned} x &= \frac{-2+14}{4} \text{ and } \frac{-2-14}{4} \\ x &= \frac{12}{4} \text{ and } \frac{-16}{4} \\ x &= 3 \text{ and } x = -4 \end{aligned}$$

Example 2:

Solve by using the quadratic formula. $-3x^2 + 21x = 30$

Before we apply the quadratic we must rewrite the equation in the form of $ax^2 + bx + c = 0$

$$\text{Now: } -3x^2 + 21x = +30$$

$$-30 \quad \neq 30$$

$$-3x^2 + 21x - 30 = 0$$

$$a = -3$$

$$b = 21$$

$$c = -30$$

$$x = 2 \text{ and } x = 5$$

$$S.S = \{2, 5\}$$

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-21 \pm \sqrt{(21)^2 - 4(-3)(-30)}}{2(-3)} \\ &= \frac{-21 \pm \sqrt{441 - 360}}{-6} \\ &= \frac{-21 \pm \sqrt{81}}{-6} \\ &= \frac{-21 \pm 9}{-6} \end{aligned}$$

$$x = \frac{-21+9}{-6} \text{ and } x = \frac{-21-9}{-6}$$

Working with Radical functions and Equations

Before we start talking about radicals, let's review square roots.

square roots are basically, numbers or expressions with rational exponent ($\frac{1}{2}$)

eg. $\sqrt{4} = (4)^{\frac{1}{2}}$

we can write 4 as 2^2

$$(4)^{\frac{1}{2}} = (2^2)^{\frac{1}{2}}$$

$$= 2^{(2 \times \frac{1}{2})}$$

$$= 2^1$$

$$= 2$$

Recall that we can multiply exponents by exponents $(x^m)^n = x^{mn}$

that is why $\sqrt{4} = 2$.

more example:

$$\sqrt{49} = (49)^{\frac{1}{2}}$$

$$49 = 7 \times 7 = 7^2$$

$$= (7^2)^{\frac{1}{2}}$$

$$= 7^{(2 \times \frac{1}{2})}$$

$$= 7^1$$

$$= 7$$

⊙ the reason why square roots are $\frac{1}{2}$ rational exponent is because "squares" means "2"

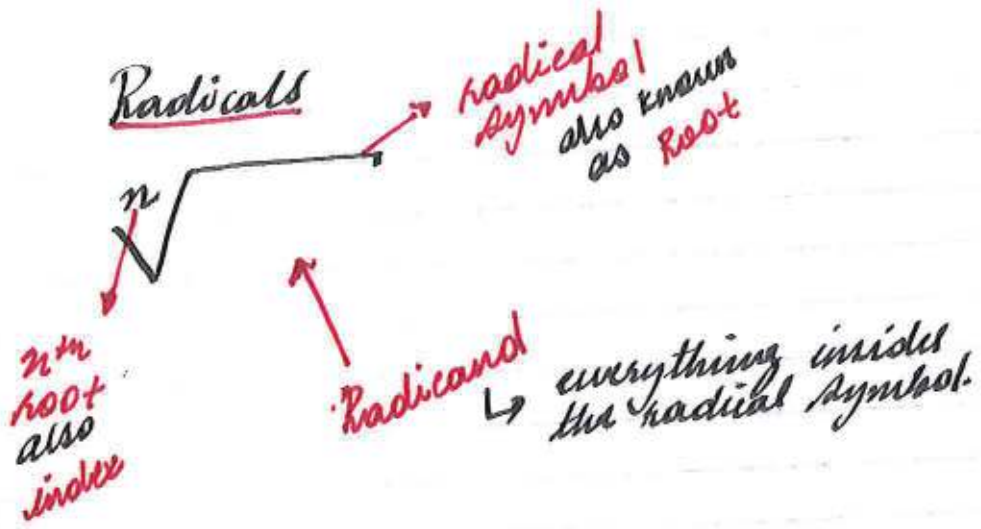
now what if we have a cube root, 4th root, 5th root and etc.

$$\sqrt[3]{8}, \sqrt[4]{x^8}, \sqrt[5]{25}, \dots$$



→ n^{th} root.

→ look for a number that can be multiplied n times by itself to give the original number.



Simplifying Radicals

Radical expressions are terms that contain radicals. i.e.

$\sqrt{x}, 2\sqrt{5}, \sqrt{3}, -3\sqrt{x+2}, \dots$

Simplify $\sqrt{18}$

If we can find a pair of numbers that are the factors of 18, then we can take one of them out. i.e.



$18 = 2 \times 3 \times 3$

$\sqrt{18} = \sqrt{2 \times 3 \times 3}$ a pair, so we can take out one of them.

$\boxed{\sqrt{18} = 3\sqrt{2}}$

Remark:

If it was a cube root we would be looking for triple factors instead of pairs.

Eg. $\sqrt{72} = ?$

$72 = 2 \times 2 \times 2 \times 3 \times 3$



$\sqrt{72} = \sqrt{2 \times 2 \times 2 \times 3 \times 3}$

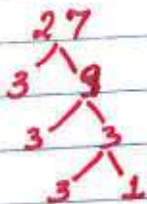
$= 2 \times 3 \sqrt{2}$

$= 6\sqrt{2}$

$\sqrt{72} = 6\sqrt{2}$

Eg. $\sqrt[3]{27}$

$27 = 3 \times 3 \times 3$



$\sqrt[3]{27} = \sqrt[3]{3 \times 3 \times 3}$

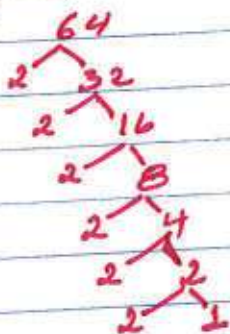
$= 3$

$\sqrt[3]{27} = 3$

since it is a cube root, we are looking for triplets.

Eg. $\sqrt[5]{64}$

$64 = 2 \times 2 \times 2 \times 2 \times 2 \times 2$



$\sqrt[5]{64} = \sqrt[5]{2 \times 2 \times 2 \times 2 \times 2 \times 2}$

$= 2 \sqrt[5]{2}$

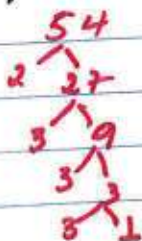
$\sqrt[5]{64} = 2 \sqrt[5]{2}$

since it is the 5th root, we are looking for five of a kind

Eg. $\sqrt[3]{54x^3y^4}$

$x^3 = x \times x \times x$

$y^4 = y \times y \times y \times y$



$\sqrt[3]{54x^3y^4} = \sqrt[3]{2 \times 3 \times 3 \times 3 \times x \times x \times x \times y \cdot y \cdot y \cdot y}$

$\sqrt[3]{54x^3y^4} = 3xy \sqrt[3]{2y}$

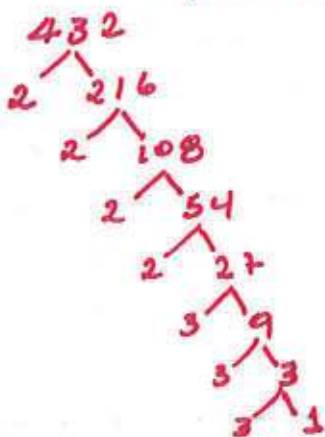
Multiplying/Dividing Radicals

Now that we know how to simplify radicals, we can start learning how to multiply and divide them.

Eg. $\sqrt{24} * \sqrt{18}$

$$\frac{\sqrt{24 * 18}}{\sqrt{432}}$$

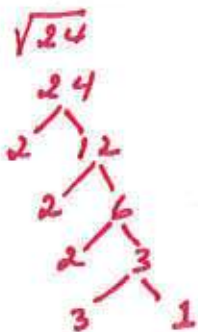
now simplify



$$\begin{aligned} \sqrt{432} &= \sqrt{2 * 2 * 2 * 2 * 3 * 3 * 3} \\ &= 2 * 2 * 3 \sqrt{3} \\ &= 12\sqrt{3} \end{aligned}$$

since both terms are square roots, we can combine them under one square root.

we also could simplify each radical before combining them under one radical. i.e.



$$\begin{aligned} \sqrt{24} &= \sqrt{2 * 2 * 2 * 3} \\ &= 2\sqrt{6} \\ \sqrt{18} &= \sqrt{2 * 3 * 3} \\ &= 3\sqrt{2} \\ \sqrt{24} * \sqrt{18} &= 2\sqrt{6} * 3\sqrt{2} \\ &= (2 * 3) \sqrt{6 * 2} \\ &= 6 \sqrt{3 * 2 * 2} \\ &= 6 * 2 \sqrt{3} \\ &= 12\sqrt{3} \end{aligned}$$

Eg:

$$\frac{\sqrt{24}}{\sqrt{6}}$$

we can combine each radical under one radical by division

$$= \sqrt{\frac{24}{6}} = \sqrt{4} = \boxed{2}$$

Eg: $\frac{\sqrt{48} \times \sqrt{12}}{\sqrt{72}}$

OR: $\sqrt{48} = 4\sqrt{3}$

$\sqrt{12} = 2\sqrt{3}$

$\sqrt{72} = 6\sqrt{2}$

$= \frac{\sqrt{48 \times 12}}{\sqrt{72}}$

$\Rightarrow \frac{\sqrt{48} \times \sqrt{12}}{\sqrt{72}} = \frac{4\sqrt{3} \times 2\sqrt{3}}{6\sqrt{2}}$

$= \frac{\sqrt{576}}{\sqrt{72}}$

$= \frac{24\sqrt{3 \cdot 3}}{6\sqrt{2}}$

$= \frac{\sqrt{576}}{\sqrt{72}}$

$= \frac{8\sqrt{3}}{\sqrt{2}}$

$= \sqrt{8} = \sqrt{2 \times 2 \times 2} = 2\sqrt{2}$

$= \frac{8\sqrt{2}}{2} = \underline{\underline{4\sqrt{2}}}$

Eg: Rationalizing the Denominator.

sometimes some denominators

are irrational. The process of

changing irrational denominators

to rational is called Rationalizing the denominator.

$\frac{2}{\sqrt{3}}$

to change the denominator to rational, multiply both the numerator and denominator by $\sqrt{3}$,

i.e.

$$\frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{\sqrt{3 \cdot 3}} = \frac{2\sqrt{3}}{\sqrt{9}} = \boxed{\frac{2\sqrt{3}}{3}}$$

Adding / subtracting Radicals.

Unlike multiplying and dividing radicals, adding and subtract can NOT be combined under one radical.

To add or subtract radicals, 1st we must simplify and see if the terms have the same values under the radical sign.

$$\text{Eq: } 2\sqrt{3} + 4\sqrt{2}$$

we can not add this two radicals, since $\sqrt{3} \neq \sqrt{2}$

$$\text{Eq: } 2\sqrt{3} - 4\sqrt{3}$$

we can take one of the $\sqrt{3}$ and combine their coefficients accordingly.

$$= (2-4)\sqrt{3}$$
$$= \boxed{-2\sqrt{3}}$$

$$\text{Eq: } 5\sqrt{2} + 2\sqrt{2} - 3\sqrt{2}$$
$$= (5+2-3)\sqrt{2}$$
$$= \boxed{4\sqrt{2}}$$

$$\text{Eq: } \sqrt{48} + \sqrt{27} - 6\sqrt{3}$$

before we decide that terms don't have a common radicand, let's simplify them

$$\sqrt{48} = 4\sqrt{3}$$
$$\sqrt{27} = 3\sqrt{3}$$

$$\sqrt{48} + \sqrt{27} - 6\sqrt{3} = 4\sqrt{3} + 3\sqrt{3} - 6\sqrt{3}$$
$$= (4+3-6)\sqrt{3}$$
$$= \boxed{1\sqrt{3}}$$

Conjugates of Radicals.

Conjugates are the sum and difference of the same radicals.

i.e. $2 + \sqrt{3}$

its conjugate is $2 - \sqrt{3}$

$2\sqrt{3} - \sqrt{2}$

its conjugate is $2\sqrt{3} + \sqrt{2}$

Multiplying two conjugates will always give us a real number.

i.e. $(2 + \sqrt{3})(2 - \sqrt{3})$

$$\begin{aligned} & 2(2) - 2(\sqrt{3}) + 2(\sqrt{3}) + \sqrt{3 \times 3} \\ & 4 - 2\sqrt{3} + 2\sqrt{3} - 3 \\ & 4 - 3 \\ & = 1 \end{aligned}$$

Eg: Rationalize the denominator

$$\frac{4}{\sqrt{5} - \sqrt{2}}$$

To get a rational denominator we need to multiply both the numerator & denominator by the denominator's conjugate, which is $\sqrt{5} + \sqrt{2}$.

$$= \frac{4}{\sqrt{5} - \sqrt{2}} \times \frac{\sqrt{5} + \sqrt{2}}{\sqrt{5} + \sqrt{2}}$$

$$= \frac{4(\sqrt{5} + \sqrt{2})}{(\sqrt{5} - \sqrt{2})(\sqrt{5} + \sqrt{2})}$$

$$= \frac{4\sqrt{5} + 4\sqrt{2}}{5 + \sqrt{10} - \sqrt{10} - 2} = \frac{4\sqrt{5} + 4\sqrt{2}}{5 - 2} = \boxed{\frac{4\sqrt{5} + 4\sqrt{2}}{3}}$$

Solving Radical Equations.

A Radical equation is an equation involving a variable under the radical sign i.e. radicand.

It is very important to talk about "Domain" when we solve radical equations. This is because, under real numbers, we can not solve negative square roots.

The domain of a radical equation is all positive numbers including zero.

Find the domain for the following:

a) \sqrt{x} ; $D = \{x \geq 0\}$

b) $\sqrt{x-2}$;

$$x-2 \geq 0$$

$$+2 \quad +2$$

$$x \geq 2$$

$$D: \{x \geq 2\}$$

c) $\sqrt{5-x}$

$$5-x \geq 0$$

$$-5 \quad -5$$

$$-x \geq -5$$

$$x-1 \quad x-1$$

$$x \leq 5$$

$$D: \{x \leq 5\}$$

notice the sign of the inequality switches when we multiply by a negative number.

After we find the domain, solving radicals is getting the variable on one side of the equation by apply the appropriate algebraic techniques.

Eg:

Solve:

$$\sqrt{x} + 5 = 8$$

$$D: \{x \in \mathbb{R} \geq 0\}$$

$$\sqrt{x} = 3$$

$$(\sqrt{x})^2 = (3)^2$$

$$x = 9$$

square both sides

the solution is part of the domain.

Eg:

$$2\sqrt{x} + 2 = 6$$

$$2\sqrt{x} + 2 = 6$$

$$-2 \quad -2$$

$$\frac{2\sqrt{x}}{2} = \frac{4}{2}$$

$$\sqrt{x} = 2$$

$$(\sqrt{x})^2 = (2)^2$$

$$x = 4$$

square both sides

$$D: \{x \in \mathbb{R} \geq 0\}$$

Eg:

$$\sqrt{2x+1} = 5$$

$$(\sqrt{2x+1})^2 = (5)^2$$

$$2x+1 = 25$$

$$-1 \quad -1$$

$$\frac{2x}{2} = \frac{24}{2}$$

$$x = 12$$

Domain:

$$2x+1 \geq 0$$

$$-1 \quad -1$$

$$\frac{2x}{2} \geq \frac{-1}{2}$$

$$x \geq -\frac{1}{2}$$

$$D: \{x \in \mathbb{R} \geq -\frac{1}{2}\}$$

check:

$$\sqrt{2(12)+1} = 5$$

$$\sqrt{24+1} = 5$$

$$\sqrt{25} = 5 \quad \checkmark$$

Eg: $\sqrt{2x+5} = \sqrt{2-x}$

$$(\sqrt{2x+5})^2 = (\sqrt{2-x})^2$$

$$2x+5 = 2-x$$

$$3x+5 = 2$$

$$\frac{3x}{3} = \frac{-3}{3}$$

$$x = -1$$

the solution is inside the domain.

Domain:

$$2x+5 \geq 0 \quad \text{And} \quad 2-x \geq 0$$

$$\frac{2x}{2} \geq \frac{-5}{2}$$

$$x \geq -5/2$$

$$-x \geq -2$$

$$x \leq 2$$



Eg:

$$\sqrt{5x} + 12 = 2$$

$$\sqrt{5x} + 12 = 2$$

$$(\sqrt{5x})^2 = (-10)^2$$

$$\frac{5x}{5} = \frac{100}{5}$$

$$x = 20$$

Domain:

$$\frac{5x}{5} \geq 0$$

$$x \geq 0$$

Check:

$$\sqrt{5(20)} + 12 \stackrel{?}{=} 2$$

$$\sqrt{100} + 12 \stackrel{?}{=} 2$$

$$10 + 12 \stackrel{?}{=} 2$$

$$22 \neq 2$$

Not True

"No solution"

Solutions like this one

is known as **Extraneous solution**:

⇒ It is always important to check your answers.

Graphing Radical Functions

In order to draw a graph for a radical function, 1st we need to figure out the "Domain" for the function.

This is because, under \mathbb{R} , we can "NOT" have negatives under the radical sign.

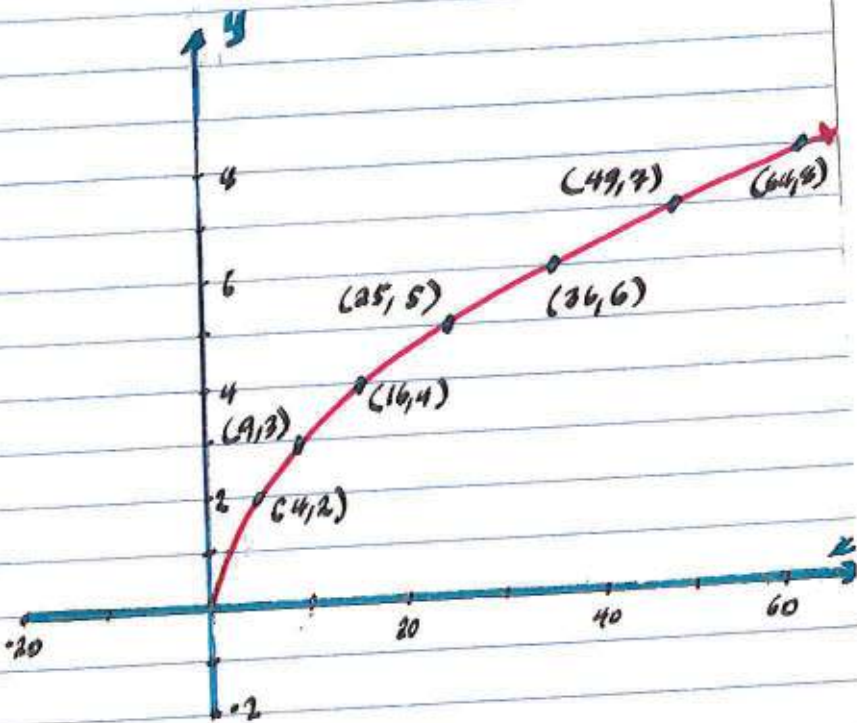
Once we know the domain, we know what kinds of numbers to put for x (input)

Eg: Graph
 $f(x) = \sqrt{x}$

Domain: $\{x \geq 0\}$

Table

x	$f(x)$	
0	0	$f(0) = \sqrt{0} = 0$
4	2	$f(4) = \sqrt{4} = 2$
9	3	$f(9) = \sqrt{9} = 3$
16	4	$f(16) = \sqrt{16} = 4$
25	5	$f(25) = \sqrt{25} = 5$
36	6	$f(36) = \sqrt{36} = 6$
49	7	$f(49) = \sqrt{49} = 7$
64	8	$f(64) = \sqrt{64} = 8$



In order to fit our table, we also scaled our x -axis.

Notice how I chose my inputs. I only chose numbers that are perfect squares.

$f(x) = \sqrt{x}$ is the parent for radical function.

The graph of other radical functions are a transformation of the graph of the parent function

Eg: $a\sqrt{x+b} + c$

↓
shrink
OR
magnify

↓
left
OR
right

↓
up
OR
down.

$f(x) = \sqrt{x-2}$ → Its graph is shifted two units to the right

$f(x) = \sqrt{x+5}$ → Its graph is shifted five units to the left.

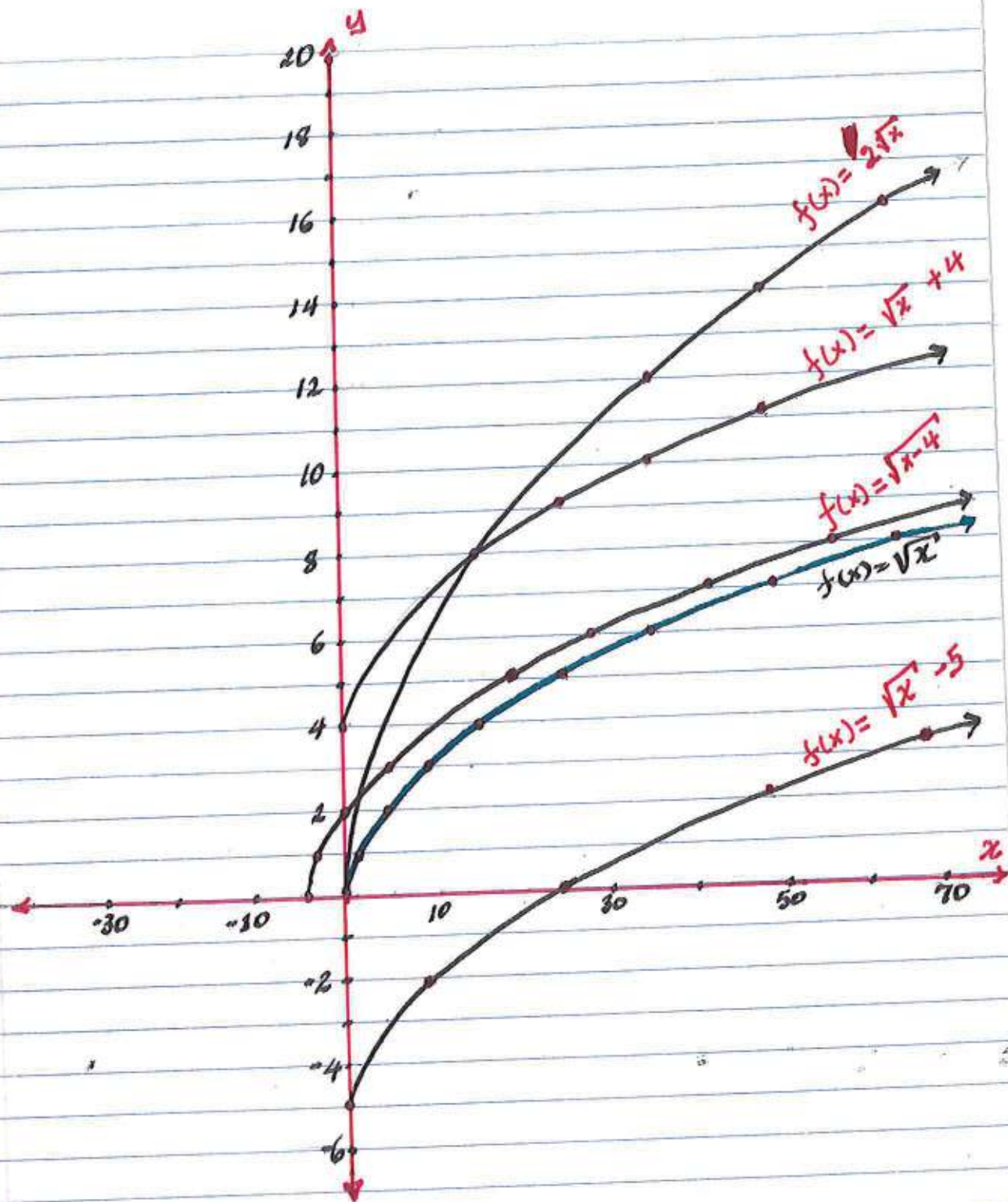
$f(x) = \sqrt{x} + 4$ Its graph is shifted four units up.

$f(x) = \frac{1}{2}\sqrt{x}$ → Its graph is magnified by 2.

$f(x) = 3\sqrt{x}$ → Its graph is shrunk by 3.

$f(x) = \frac{1}{4}\sqrt{x+1} - 2$ Its graph is shifted one unit to the left, two units down and magnified by four.

All of these functions' graphs are in comparison with its parent function $f(x) = \sqrt{x}$



The graphs of radical functions demonstrate transformation of the parent function $f(x) = \sqrt{x}$.

Working with Rational Functions and Equations.

In this section we are going to study simplifying, solving and graphing rational expressions, equations and functions respectively.

Fractions review:

It is important to go over some fraction concepts. *multiplying, dividing, adding, subtracting and simplifying.*

Eg: $\frac{12}{18} = ?$ $\frac{12}{18}$ by 2 $\frac{6}{9}$ by 3 $\frac{2}{3}$
 $\frac{12}{18} = \boxed{\frac{2}{3}}$

Eg: $\frac{14}{15} * \frac{3}{35} = ?$ $\frac{14}{15} * \frac{3}{35}$ by 3 $\frac{14}{5} * \frac{1}{35}$ by 7 $\frac{2}{5} * \frac{1}{5} = \boxed{\frac{2}{25}}$
cross simplifying

Eg: $\frac{24}{16} \div \frac{3}{4}$ we can switch division to multiplication by flipping the divisor

v.e $\frac{24}{16} * \frac{4}{3}$ by 4 $\frac{24}{4} * \frac{1}{3}$ by 3 $\frac{6}{1} * \frac{1}{3}$ by 2 $\frac{2}{1} = \boxed{2}$

Eg: $\frac{4}{3} + \frac{5}{4}$ To add or subtract fractions, we need to have a common denominator

$$\left(\frac{4}{4}\right)\frac{4}{3} + \frac{5}{4}\left(\frac{3}{3}\right) = \frac{4*4}{4*3} + \frac{5*3}{4*3} = \frac{16}{12} + \frac{15}{12} = \frac{16+15}{12} = \boxed{\frac{31}{12}}$$

Rational expressions are fractions where the numerators and denominators are polynomials.

Eg: $\frac{2}{x}$, $\frac{3x^2+1}{x^3}$, $\frac{x^2+2}{x^2-4}$, $\frac{x^2+5x+6}{x-2}$

are all rational expressions.

One of the most important things when we work with rational functions is making sure that the denominators are different from zero.

If the denominator is zero, then the function becomes undefined, because dividing by zero is not possible.

Figuring out the value of x (variable) that makes the denominator zero helps us to find the domain of the function.

The domain of a rational function is all sets of real numbers except those values that makes the denominator zero.

Eg: $\frac{1}{x-2}$

To find the domain, set the denominator to equals to zero and find the x value/values that makes it zero.

And exclude those values.

$$x-2=0$$

$$+2 \quad +2$$

$$x=2$$

$$D = \{x \in \mathbb{R} \neq 2\}$$

$$D = \{x \in \mathbb{R} / 2\}$$

Simplifying Rational Expressions.

Just like fraction, if rational expressions have common factors, we can simplify them.

Eg: $\frac{2x^3}{x}$

$$\frac{2x^2 \cdot x}{x} = 2x^2$$

Eg: $\frac{3x+6}{x+2}$

$$\frac{3x+6}{x+2} = \frac{3(x+2)}{(x+2)} \quad \text{factoring 3,}$$
$$= 3$$

Eg: $\frac{4x^2}{2x^2+4x}$

$$= \frac{4x^2}{2x^2+4x}$$
$$= \frac{4x^2}{2x(x+2)}$$
$$= \frac{\cancel{2x} \cdot 2x}{\cancel{2x}(x+2)}$$
$$= \frac{2x}{x+2}$$

we can factor out $2x$
from the denominator
and

$$4x^2 = 2x \cdot 2x$$

Eg: $\frac{x+2}{x^2-4}$

$$= \frac{x+2}{x^2-4}$$

$$= \frac{\cancel{(x+2)}}{\cancel{(x+2)}(x-2)} = \frac{1}{x-2}$$

$$x^2-4 = x^2-2^2$$
$$= (x-2)(x+2)$$

Remark:

factoring polynomials play an important role in simplifying rational expressions.

Eg: $\frac{2x^3 + 3x^2 + 8x + 12}{2x^2 - x - 6}$

Before we try to simplify this expression, let's factor both the numerator and denominator.

$$\begin{aligned} 2x^2 - x - 6 \\ 2x^2 - 4x + 3x - 6 \\ 2x(x-2) + 3(x-2) \\ (x-2)(2x+3) \end{aligned}$$

Sum.	prod.
-1	-12
#5	-4 2 3

$$\begin{aligned} 2x^3 + 3x^2 + 8x + 12 \\ \text{factoring by grouping} \\ (2x^3 + 3x^2) + (8x + 12) \\ x^2(2x+3) + 4(2x+3) \\ (2x+3)(x^2+4) \end{aligned}$$

$$\frac{2x^3 + 3x^2 + 8x + 12}{2x^2 - x - 6} =$$

$$= \frac{(2x+3)(x^2+4)}{(x-2)(2x+3)}$$

$$= \boxed{\frac{x^2+4}{x-2}}$$

we can not simplify this any further so we stop at $\frac{x^2+4}{x-2}$

as the most simplified form.

Multiplying and Dividing Rational Expression

Multiplying & dividing rational expressions are similar to multiplying and dividing fractions.

Eg: multiply $\frac{x^2-1}{x}$ by $\frac{x^2}{x-1}$

$$= \frac{(x^2-1)}{x} * \frac{x^2}{x-1}$$

$$x^2-1 = x^2-1^2 \\ = (x-1)(x+1)$$

$$x^2 = x \cdot x$$

$$= \frac{\cancel{(x-1)}(x+1)}{\cancel{x}} * \frac{\cancel{x}(x)}{\cancel{(x-1)}}$$

$$= \frac{(x+1)}{1} * \frac{x}{1}$$

$$= x^2 + x$$

Eg: multiply $\frac{x^2+5x+6}{x^2-4}$ by $\frac{x+2}{x+3}$

$$\frac{(x^2+5x+6)}{(x^2-4)} * \frac{(x+2)}{(x+3)}$$

	sum	product
x^2+5x+6	5	6
$x^2+2x+3x+6$	#'s are	2 2 3
$x(x+2)+3(x+2)$		
$(x+2)(x+3)$		

$$x^2-4 = x^2-2^2 \\ = (x-2)(x+2)$$

$$= \frac{\cancel{(x+2)}(x+3)}{\cancel{(x+2)}(x-2)} * \frac{(x+2)}{(x+3)}$$

$$= \frac{\cancel{(x+3)}(x+2)}{(x-2)\cancel{(x+3)}}$$

$$= \boxed{\frac{(x+2)}{(x-2)}}$$

* Remark: Always make sure that you have simplified the rational expression to their simplified form.

Ex: Divide $\frac{x^3+8}{x-2}$ by $\frac{x+2}{x-2}$

$\frac{x^3+8}{x-2} \div \frac{x+2}{x-2}$, we can convert the division into multiplication by flipping the divisor. i.e.

$$\Rightarrow \frac{x^3+8}{x-2} \times \frac{x-2}{x+2}$$

$$\Rightarrow \frac{x^3+8}{x+2}$$

$$\Rightarrow \frac{(x+2)(x^2+2x+4)}{(x+2)}$$

$$\boxed{x^2-2x+4}$$

$$x^3+8 = x^3+2^3 = (x+2)(x^2-2x+4)$$

Ex: Divide $\frac{x^2-6x+8}{2x+5}$ by $\frac{2x^2+x-10}{x^2+1}$

$$\frac{x^2-6x+8}{2x+5} \div \frac{2x^2+x-10}{x^2+1}$$

$$\Rightarrow \frac{x^2-6x+8}{(2x+5)} \times \frac{(x^2+1)}{2x^2+x-10}$$

$$\Rightarrow \frac{(x-2)(x-4)}{(2x+5)} \times \frac{(x^2+1)}{(x-2)(2x+5)}$$

$$\Rightarrow \frac{(x-4)(x^2+1)}{(2x+5)(2x+5)}$$

$$\Rightarrow \frac{x^3-4x^2+x-4}{(2x+5)^2}$$

$$\begin{aligned} x^2-6x+8 \\ x^2-2x-4x+8 \\ x(x-2)-4(x-2) \\ (x-2)(x-4) \end{aligned}$$

$$\begin{aligned} 2x^2+x-10 \\ 2x^2-4x+5x-10 \\ 2x(x-2)+5(x-2) \\ (x-2)(2x+5) \end{aligned}$$

$$\boxed{\frac{x^3-4x^2+x-4}{4x^2+20x+25}}$$

Adding and subtracting Rational expressions:

when we add or subtract rational expression, we need to make sure that they have a common denominator.

Technique:

To get the same denominator, check what each denominator is missing to be the same as the other denominators, then multiply both the numerators and denominators by the missing expression.

Eg: $\frac{1}{2x} + \frac{1}{x^2}$

$$\left(\frac{x}{x}\right) \frac{1}{2x} + \frac{1}{x^2} \left(\frac{2}{2}\right)$$

$$\frac{x \cdot 1}{x \cdot 2x} + \frac{1 \cdot 2}{x^2 \cdot 2}$$

$$\frac{x}{2x^2} + \frac{2}{2x^2}$$

$$\frac{x+2}{2x^2}$$

since we can no longer simplify, our answer is

$$\boxed{\frac{x+2}{2x^2}}$$

compare $2x$ and x^2
 $2x$ needs one more x
 x^2 needs 2.

so multiply $\frac{1}{2x}$ by $\frac{x}{x}$ and $\frac{1}{x^2}$ by $\frac{2}{2}$

now each fraction expression has the same denominator, we can combine / add them

Eg: add $\frac{x^2+x}{x+1}$ and $\frac{2}{x}$

$$\frac{x^2+x}{x+1} + \frac{2}{x}$$

multiply $\frac{x^2+x}{(x+1)}$ by $\frac{x}{x}$ and

multiply $\frac{2}{x}$ by $\frac{x+1}{x+1}$

$$\left(\frac{x}{x}\right) \frac{x^2+x}{x+1} + \frac{2}{x} \left(\frac{x+1}{x+1}\right)$$

$$\frac{x(x^2+x)}{x(x+1)} + \frac{2(x+1)}{x(x+1)}$$

$$\frac{x^3+x^2}{x^2+x} + \frac{2x+2}{x^2+1}$$

- Now that they have the same denominator, we can add the numerators.

$$\frac{x^3+x^2+2x+2}{x^2+x}$$

$$\frac{\cancel{(x+1)}(x^2+1)}{x\cancel{(x+1)}}$$

$$\frac{x^3+x^2+2x+2}{(x+1)(x^2+1)}$$

$$\frac{x^2+x}{x(x+1)}$$

$$\boxed{\frac{x^2+1}{x}}$$

In this problem we were able to simplify the rational expressions after adding them. This is because both the numerator and denominator have a common factor i.e. $(x+1)$

Subtraction:

Eg. Subtract $\frac{2x}{x+1}$ from $\frac{2x^2}{x^2+4x+3}$

$$\frac{2x^2}{x^2+4x+3} - \frac{2x}{x+1}$$

$$\begin{array}{r} x^2+4x+3 \quad \underline{5} \quad \underline{P} \\ x^2+1x+3x+3 \quad \underline{4} \quad \underline{S} \\ x(x+1)+3(x+1) \quad \text{\#}'s \quad 3, 2, 2 \\ (x+1)(x+3) \end{array}$$

1st we need to factor x^2+4x+3 in order to see by what expression we need to multiply to get the same denominator.

$$\frac{2x^2}{x^2+4x+3} - \frac{2x}{x+1}$$

we just need to multiply the 2nd expression by $\frac{x+3}{x+3}$

$$\frac{2x^2}{(x+1)(x+3)} - \frac{2x}{(x+1)} \left(\frac{x+3}{x+3} \right)$$

$$\frac{2x^2}{(x+1)(x+3)} - \frac{2x(x+3)}{(x+1)(x+3)}$$

$$\frac{2x^2}{(x+1)(x+3)} - \frac{2x^2+6x}{(x+1)(x+3)}$$

Now we can subtract the numerators.

$$\frac{2x^2 - (2x^2+6x)}{(x+1)(x+3)}$$

$$\frac{2x^2 - 2x^2 - 6x}{(x+1)(x+3)} \rightarrow \frac{-6x}{x^2+4x+3}$$

$$\boxed{\frac{-6x}{x^2+4x+3}}$$

Solving Rational Equation:

Whenever we solve rational equations, we are looking for values of the variables that will make the equation true.

Sometimes we can not find any solution and sometimes we can find more than solution.

Before we solve rational functions / Equations we need to determine the "Domain"

In this way, if we find solutions that are not in the domain, we can eliminate them.

Eg: $\frac{x-3}{2} = \frac{1}{x}$

Domain $\{ \mathbb{R} \mid x \neq 0 \}$ x can not be zero.

$$\frac{x-3}{2} = \frac{1}{x}$$

now lets make the denominators the same.

$$\left(\frac{x}{x}\right) \frac{x-3}{2} = \frac{1}{x} \left(\frac{2}{2}\right)$$

$$\frac{x(x-3)}{2x} = \frac{2}{2x}$$

$$x^2 - 3x = 2$$

+2 +2

for $x \neq 0$, we can eliminate the denominators

$$x^2 - 3x + 2 = 0$$

$$x^2 - 2x - 1x + 2 = 0 \quad \begin{array}{l} \underline{S} \\ -3 \end{array} \quad \begin{array}{l} \underline{P} \\ 2 \end{array}$$

$$x(x-2) - 1(x-2) = 0 \quad \begin{array}{l} \#15 \\ -2x-1 \end{array}$$

$$(x-2)(x-1) = 0$$

$$x-2=0 \quad \text{or} \quad x-1=0$$

$$x=2$$

$$x=1$$

$$S.S = \{ x=1, 2 \}$$

Eg:

Solve $\frac{1}{x-4} + \frac{1}{x+5} = \frac{9}{x^2+x-20}$

1st let's add the left side after finding the domain.

Domain $x-4 \neq 0$ $x+5 \neq 0$
 $x \neq 4$ $x \neq -5$

$$\left(\frac{x+5}{x+5}\right) \frac{1}{x-4} + \frac{1}{x+5} \left(\frac{x-4}{x-4}\right) = \frac{9}{x^2+x-20}$$

$$\frac{x+5}{(x+5)(x-4)} + \frac{x-4}{(x+5)(x-4)} = \frac{9}{x^2+x-20}$$

↓

$$x^2+x-20 \quad \begin{matrix} 5 & 1 \\ 1 & -20 \end{matrix}$$

$$x(x-4)+5(x-4) \neq 5 \quad \text{B \& -4}$$

$$\frac{x+5}{(x+5)(x-4)} + \frac{x-4}{(x+5)(x-4)} = \frac{9}{(x+5)(x-4)}$$

$$\frac{x+5+x-4}{(x+5)(x-4)} = \frac{9}{(x+5)(x-4)}$$

$$\frac{2x+1}{(x+5)(x-4)} = \frac{9}{(x+5)(x-4)}$$

For $x \neq -5 \& 4$, we can cancel the denominators.

$$2x+1 = 9$$

$$2x = 8$$

$$x = 4$$

But $x=4$ is not in our domain, as a result there is No solution.

Extraneous Solution

Eg:
solve

$$\frac{2x^2+x-1}{x^2+x-6} = 0$$
$$\frac{1}{x-2} + \frac{1}{x+3}$$

Domain:
we have 3 individual
fractions.

1. $x^2+x-6 \neq 0$
2. $x-2 \neq 0$
3. $x+3 \neq 0$

lets factor $2x^2+x-1$ and
 x^2+x-6

$$\Rightarrow D: \{R \mid x \neq -3, 2\}$$

$2x^2+x-1$	x^2+x-6
$\begin{matrix} \Sigma & P \\ -1 & -2 \\ \neq -2, 1 \end{matrix}$	$\begin{matrix} \Sigma & P \\ 1 & -6 \\ \neq 3, -2 \end{matrix}$
$2x^2-2x+1x-1$	$x^2+3x-2x-6$
$2x(x-1)+1(x-1)$	$x(x+3)-2(x+3)$
$(x-1)(2x+1)$	$(x+3)(x-2)$

$$\frac{2x^2+x-1}{x^2+x-6} = 0 \quad \frac{(x-1)(2x+1)}{(x+3)(x-2)} = 0$$
$$\frac{1}{x-2} + \frac{1}{x+3} = \left(\frac{x+3}{x+3}\right)\frac{1}{x-2} + \frac{1}{x+3}\left(\frac{x-2}{x-2}\right)$$
$$= \frac{(x-1)(2x+1)}{(x+3)(x-2)} = 0$$
$$\frac{x+3}{(x+3)(x-2)} + \frac{x-2}{(x+3)(x-2)}$$

$$= \frac{(x-2)(2x+1)}{(x+3)(x-2)} = 0 \quad \text{for } x \neq -3 \text{ \& } 2$$
$$\frac{x+3+x-2}{(x+3)(x-2)}$$

$$= \frac{(x-2)(2x+1)}{(2x+1)} = 0 \quad \text{for } x \neq -\frac{1}{2}$$

Extraneous
solution

$$x-2=0$$

$$x=2$$

Ans

No solution

Graphs of Rational functions:

Rational functions are one of these functions, where their domain have some restrictions.

This implies that their graphs are not continuous through out the real numbers

These restrictions are values of x 's which makes the denominator zero/undefined and horizontal lines which the graph can not cross.

Defn. Lines where ~~the~~ a rational function graph gets closer and closer but never cross or touch are called Asymptotes.

There are two types of asymptotes.

1. Vertical asymptote

$$x = a, a \in \mathbb{R}$$

2. Horizontal asymptote

$$y = k, k \in \mathbb{R}$$

↳ sometimes oblique

$$y = mx + b$$

(H.A.)

(V.A.)

Vertical Asymptotes:

Any values of domain restrictions are vertical asymptotes

Horizontal asymptotes:

1. $y = 0$, if the degree of the denominator is bigger than the degree of the numerator.

2. If the degree of the numerator and denominator is the same, then $y =$ the ratio of the coefficients of the terms with the highest degree.

3. If the degree of the numerator is bigger than the denominator, then $y =$ the quotient of the numerator and denominator.

Ex:

$$f(x) = \frac{2x}{-x^3+1}$$

$$\begin{aligned} \text{V. A.} \\ -x^3+1 &= 0 \\ -1 & -1 \\ -x^3 &= -1 \\ x-1 & x-1 \\ \sqrt[3]{x^3} &= \sqrt[3]{1} \\ x &= 1 \end{aligned}$$

$$x=1$$

H.A.
since the degree of the denominator is bigger than the degree of the numerator

$$y=0$$

$$f(x) = \frac{3x-1}{2x+3}$$

$$\begin{aligned} 2x+3 &= 0 \\ -3 & -3 \\ 2x &= -3 \\ \frac{2x}{2} &= \frac{-3}{2} \\ x &= -\frac{3}{2} \end{aligned}$$

$$x = -\frac{3}{2}$$

The degrees are the same, so let's find the ratio of their coefficients

$$y = \frac{3}{2}$$

$$f(x) = \frac{x^2+5}{x}$$

$$x=0$$

This is an oblique asymptote:

$$\begin{array}{r} x \cdot \frac{x^2+5}{x} \\ \underline{-x^2} \\ -5 \end{array}$$

$$y=x$$

EX:

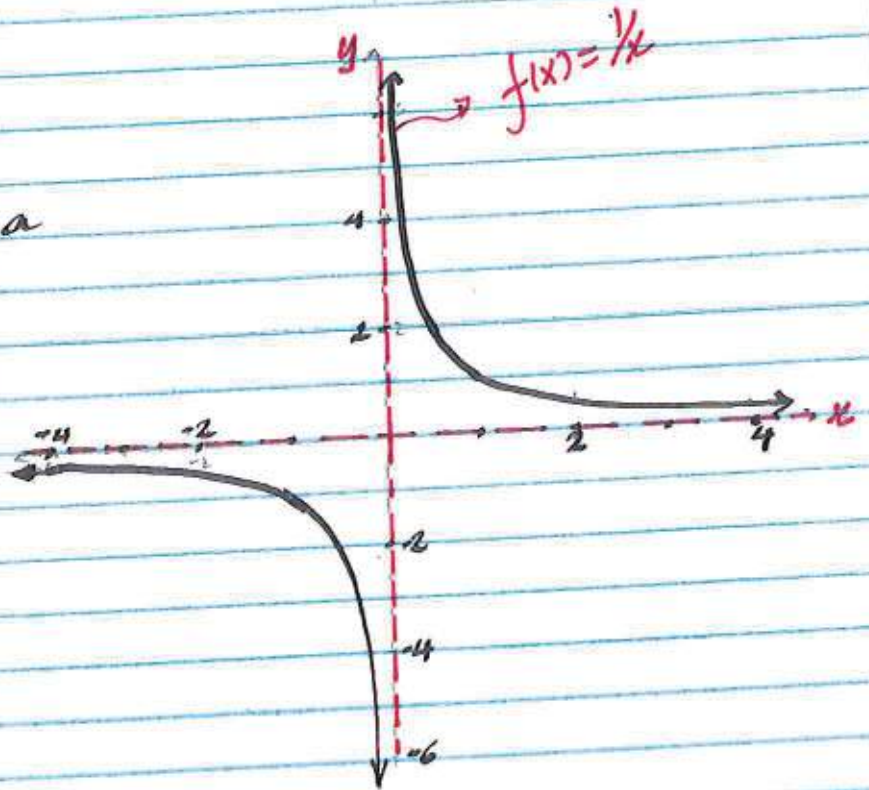
Graph
 $f(x) = \frac{1}{x}$

V.A. $x=0$

H.A. $y=0$

We could create a table.

x	$f(x)$
-4	$-\frac{1}{4}$
-2	$-\frac{1}{2}$
-1	-1
0	\exists
1	1
2	$\frac{1}{2}$
4	$\frac{1}{4}$



1st draw the V.A. & Horizontal A. by using dashed line (i.e. ----)

Notice that the graph gets closer and closer to the lines $x=0$ & $y=0$ but never touch or cross those lines.

Observation:

As x approaches to '0' from the left, the graph gets very smaller and smaller and end up approaching $-\infty$.

As x approaches to '0' from the right, the graph gets very larger and larger and end up approaching $+\infty$.

Example:

Draw the graph

of $f(x) = \frac{-2x^2}{x^2-4}$

V.A: $x^2-4=0$

$x^2-2^2=0$

$(x-2)(x+2)=0$

V.A: $x=-2, x=2$

H.A: the degrees are
the same

\Rightarrow H.A. = $y = \frac{-2}{1}$

$y = -2$

