# 1.1 Defining Functions

Functions govern many interactions in our society today. Whether buying a cup of coffee at the local coffee shop or playing a video game, we are using a function in some fashion.

## Definition of a Function

A **function** is a rule or relationship between two quantities, often referred to as the input and output, such that for every input there is exactly one output. If we input a specific value into the function, we get a specific output as an answer. We won't get the possibility of two answers or else it wouldn't be a function. The most common example of a function is an equation such as:

$$y = 2x + 3$$

In this case, $x$ is the input and $y$ is the output. If we substitute a value for $x$, say $x = 3$, then we will get an answer for $y$, namely $y = 9$, as the output. Notice that every time we input $x = 3$ you will get the output of $y = 9$. Since we always get only a single output for any value we input, this is a true function.

An example of an equation that is not a function would be $y^2 = x$. Notice that if we input $x = 4$, then $y = 2$ could be the output or $y = -2$ could be the output. Therefore this is not a true function unless we make the function $y = \sqrt{x}$ where we take only the principal (or positive) square root.

## Function Notation

Functions represented by equations have a different notation. We are used to an equation looking like this: $y = 2x + 3$. However, from this point forward function equations will be written as follows:

$$f(x) = 2x + 3$$

We would say that $f$ is a function of $x$ such that if you input $x$, you will output $2x + 3$. The advantage of this notation is that we clearly know what our input and output are. For example, we might use the function $h(t) = -16t^2 + 48t$ to represent the height of ball thrown in the air over time. We would say that the height of the ball, $h$, is a function of the time since you have thrown it in the air, $t$.

## Evaluating Functions

Evaluating a function means to figure out what the output is when given a specific input. Let's look at the following function that shows the total cost at an amusement park, $c$, depending on the number of tickets bought, $t$, to ride the rides.

$$c(t) = 2t + 3$$

We can evaluate this function for $t = 5$, or $c(5)$, by substituting into the equation as follows:

$$c(5) = 2(5) + 3 = 10 + 3 = 13$$

So our output is $c(5) = 13$ meaning it the cost to buy $5$ tickets is $\$13$. Let's evaluate the same function for $10$ tickets by looking for $c(10)$.

$$c(10) = 2(10) + 3 = 20 + 3 = 23$$

Our output this time is $c(10) = 23$ meaning the cost to buy $10$ tickets is $\$23$.

## Domain of a Function

The set of all possible inputs is called the **domain** of a function. For $f(x) = 2x + 3$ the domain is all real numbers. Any number we want could be input into the function as $x$. One way we can write this is in set notation as follows:

$$D: (-\infty, \infty)$$

This means that the domain of the function is any number between negative infinity and infinity. So we can input any number we want for $x$. However, in the context of the carnival as described above, it would only make sense to think about the domain $D: [0, \infty)$ since we wouldn't buy negative tickets.

Notice the bracket $[$ instead of the parentheses. The bracket means that it can equal that number. When we use the parentheses, we mean it cannot actually equal the number. So if we can input zero and above (greater than or equal to zero) we use the domain $D: [0, \infty)$, but if we can only input numbers strictly greater than zero, we use the domain $D: (0, \infty)$.

Some functions have limited domains or ranges. For example, in the function $f(x) = \sqrt{x + 5}$ we can only input $x \geq -5$ because we can't take the square root of a negative. This means the domain would be written as $D: [-5, \infty)$.

Another example of a limited domain is the function $f(x) = \frac{100}{x}$ which has a domain of any number except $0$ (since we can't divide by $0$). We might write this out by saying the domain is $D: x \neq 0$. Note that this is not in set notation, but rather written as an inequality. There is nothing wrong with using the most efficient method to communicate the domain.

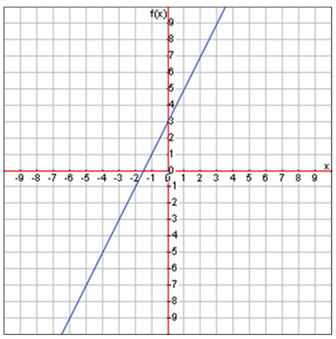
## Range of a Function

Similar to the domain, all of the outputs we could possibly get are called the **range**. More precisely, the range is the set of all possible outputs for a function. The range for the function $f(x) = 2x + 3$ is all real numbers. This means the output, or $f(x)$ value, could be any number and would write the range of this function as $R: (-\infty, \infty)$.

Consider the function $f(x) = x^2 - 2$. Notice that no matter what value we plug in for $x$, we will always output a number greater than or equal to $-2$. Therefore we would write this range as $R: [-2, \infty)$.

It is often easier to see the domain and range from the graph of a function. Let's consider a few examples.

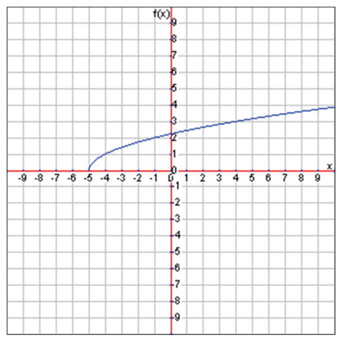
$f(x) = 2x + 3$



Domain: Any $x$ because the graph continues forever both left and right. $D: (-\infty, \infty)$

Range: Any output because the graph continues forever both up and down. $R: (-\infty, \infty)$

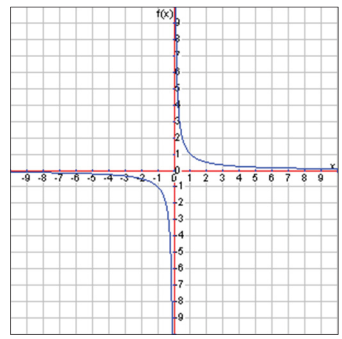
$f(x) = \sqrt{x + 5}$



Domain: The inputs must be greater than or equal to $-5$ because the graph continues forever to the right starting at $-5$. $D: [-5, \infty)$

Range: All outputs will be greater than or equal to $0$ because the graph continues forever up from the height of $0$. $R: [0, \infty)$

$f(x) = \frac{1}{x}$



Domain: $x \neq 0$ because the graph continues forever to the left and right but never touches $x = 0$. $D: x \neq 0$

Range: The graph continues forever up and down but never has a height of $y = 0$. $R: f(x) \neq 0$

## Is it a Function?

So how exactly can we tell if something is a function? The definition is that every input has only one output. This means that not only equations can be functions, but graphs, tables, and words can be functions. For example consider the following examples of potential functions.

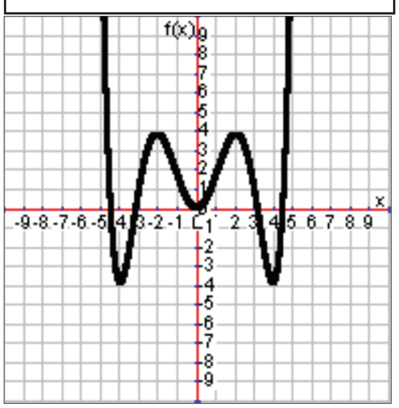
| Example 1 | Example 2 |
| --- | --- |
| Input: A person’s identity<br>Output: Their social security number | Input: A person’s age<br>Output: Their weekly income |

In this first example, if we input someone’s identity, say your math teacher, will we output only one social security number? Yes. One person never has two social security numbers. So example one is a true function. The domain of the function would be any citizen of the USA (since social security is our thing) and the range would be all the social security numbers.

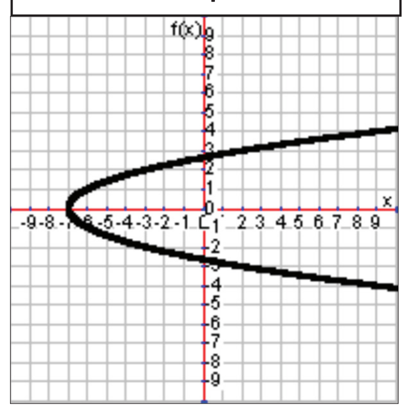
In the second example, if we input someone’s age, say $33$ years old, will we output only one weekly income? No. One thirty-three year old could make $\$200$ per week and another could make $\$1{,}000$ per week. That is two outputs for one input, so it is not a function. Since it’s not a function, we don’t have to worry about the domain and range.

Sometimes it’s easier to see whether something is a function in graph form. Consider the following two graphs and decide if they are functions or not.

**Example 3**




**Example 4**




Here we can use what is called the vertical line test. Since the inputs are the $x$ values, if for any $x$ value at all we get multiple outputs (or $y$ values on the graph), then it is not a function. Take a pencil and lay it down vertically (up and down) on the left side of Example 3. Now slowly push the pencil to the right. Is there any place where the pencil touches the graph more than once? No. That means every input has only one output, so it is a function. Any input looks fine and the graph is always higher than $-4$. Therefore the domain is $D: (-\infty, \infty)$ and the range is approximately $R: (-4, \infty)$ since it looks like it doesn’t quite touch the height of $-4$.

If you do the same thing on Example 4, the pencil will hit the graph in two places starting to the right of $x = -7$. One particular example is look at $x = -3$. In that case we have outputs of both $y = 2$ and $y = -2$. With two outputs, the vertical line hitting twice, this cannot be a function.