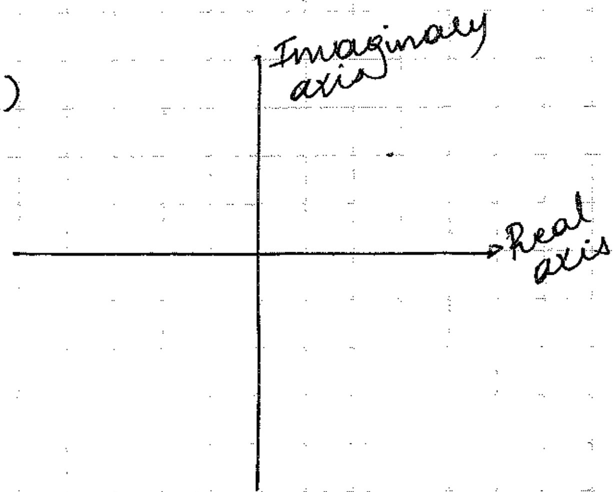
Complex Analysis.

picturing the complex numbers.

$z = x + iy$
$\rightsquigarrow (x, y)$

complex plane.

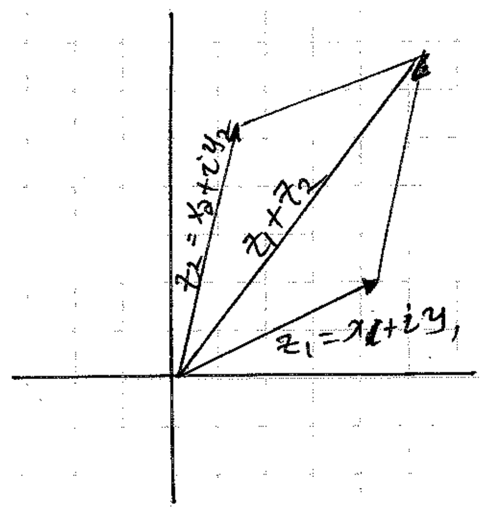


The modules or length of complex numbers $\mathbb{C}$, $z = x + iy$ is $\sqrt{x^2+y^2}$

If $z = x + yi$
$\bar{z} = x - yi$

$$z\bar{z} = x^2 + y^2$$

$\Rightarrow$ Modules of $z = |z| = \sqrt{z\bar{z}}$

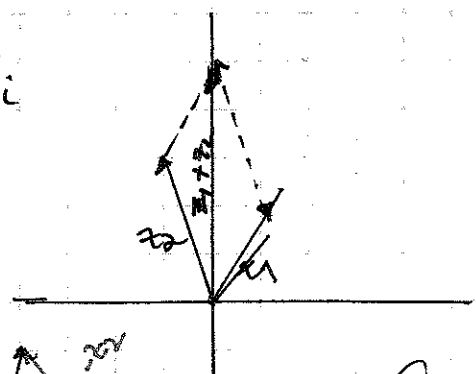


Geometry of Addition.

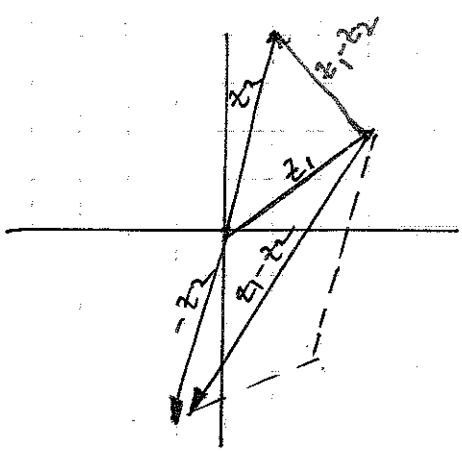
if. $z_1 = x_1 + y_1 i$
$z_2 = x_2 + y_2 i$

know $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$

EX: $z_1 = 1 + 2i$
$z_2 = -1 + 3i$
$z_1 + z_2 = 5i$



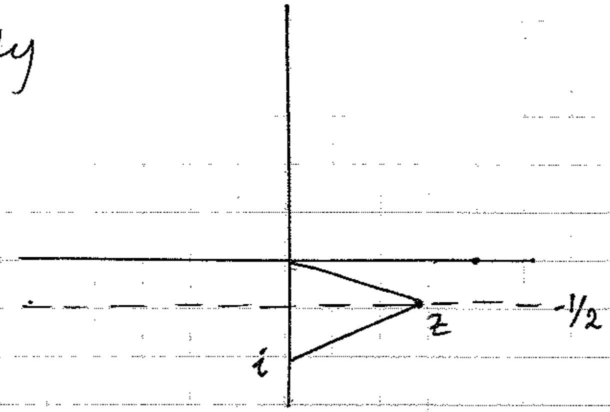
what is $z_1 - z_2$



Conclusion
$|z_1 - z_2|$ gives the distance b/n z_1 & z_2

$d = |z_1 - z_2|$
$= |(1+2i) - (-1+3i)|$
$= |2 - i|$
$= \sqrt{5}$

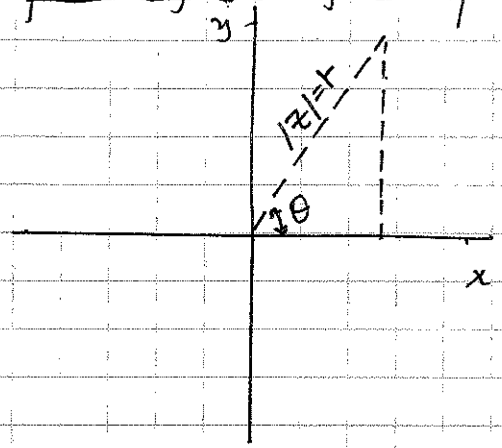
Describe the set of points which satisfy $|z| = |z+i|$



let $z = x + yi$

$$|x+yi| = |x+yi+i|$$
$$\sqrt{x^2+y^2} = |x+(y+1)i|$$
$$\sqrt{x^2+y^2} = \sqrt{x^2+(y+1)^2}$$
$$\cancel{x^2}+\cancel{y^2} = \cancel{x^2}+\cancel{y^2}+2y+1$$
$$\underline{\underline{y = -1/2}}$$

Polar form of Complex Numbers.



$$z = x + iy$$
$$(x, y)$$
$$\cos\theta = x/r$$
$$x = r\cos\theta$$
$$\sin\theta = y/r$$
$$y = r\sin\theta$$
$$z = x + iy$$
$$= r\cos\theta + i\,r\sin\theta$$
$$= r(\cos\theta + i\sin\theta) \longrightarrow \text{Polar form of } z$$

$r = |z|$

θ = argument of z

$= \arg z$

Ex. Write $z = -2i$ in polar form

$$= 0 - 2i$$
$$r = |z| = \sqrt{4} = 2$$
$$\sin\theta = y/r$$
$$\theta = \sin^{-1}(-2/2) = \sin^{-1}(-1)$$
$$= 3\pi/2 = -\pi/2$$
$$z = r(\cos\theta + i\sin\theta)$$
$$= 2(\cos(-\pi/2) + i\sin(-\pi/2))$$

Arg z is not unique!

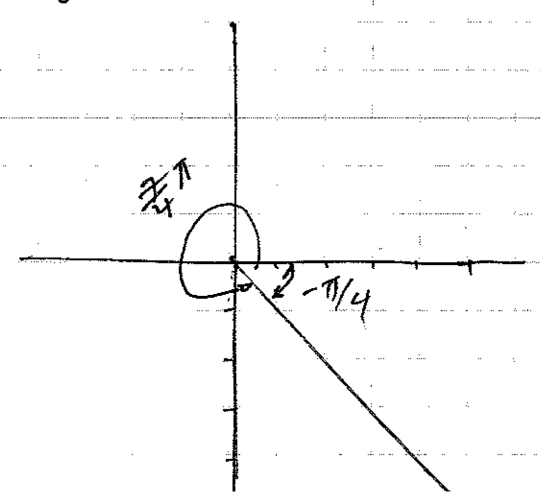
If we choose $-\pi < \theta \le \pi$, th

θ is the principle value of z and write Arg z.

$\text{Arg}\, z = -\pi/2$

write $z = 5-5i$ in polar form.

$$r = \sqrt{5^2+(-5)^2} = \sqrt{50}$$
$$= 5\sqrt{2}$$
$$\sin\theta = y/r \Rightarrow \theta = \sin^{-1}(y/r)$$
$$= \sin^{-1}(-5/5\sqrt{2})$$
$$= \sin^{-1}(-\tfrac{\sqrt{2}}{2})$$
$$= -\pi/4 = 7\pi/4$$



$$z = 5\sqrt{2}(\cos(-\pi/4) + i\sin(-\pi/4))$$

$\text{Arg}\, z = -\pi/4, \quad -\pi < \theta < \pi$

$z_1 = r_1[\cos(\theta_1) + i\sin(\theta_1)]$

$z_2 = r_2[\cos(\theta_2) + i\sin(\theta_2)]$

$$\begin{aligned} z_1 z_2 &= r_1 r_2 [\cos(\theta_1) + i\sin(\theta_1)][\cos(\theta_2) + i\sin(\theta_2)] \\ &= r_1 r_2 [\cos\theta_1\cos\theta_2 + \underline{\cos\theta_1\sin\theta_2 i} + \underline{\sin\theta_1\cos\theta_2 i} - \sin\theta_1\sin\theta_2] \\ &= r_1 r_2 [\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)] \end{aligned}$$

Ex. $z_1 = \sqrt{2}\,[\cos(\pi/4) + i\sin(\pi/4)]$

$z_2 = \sqrt{3}\,[\cos(\pi/12) + i\sin(\pi/12)]$

$r_1 = \sqrt{2}$ $\quad \theta_1 = \pi/4$ $\quad , \theta_1 + \theta_2 = \pi/4 + \pi/12$

$r_2 = \sqrt{3}$ $\quad \theta_2 = \pi/12$ $\quad = \pi/3$

$r_1 r_2 = \sqrt{6}$

$z_1 z_2 = \sqrt{6}\,[\cos(\pi/3) + i\sin(\pi/3)]$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]$$

$$z^2 = r^2[\cos(2\theta) + i\sin(2\theta)]$$
$$z^3 = r^3[\cos(3\theta) + i\sin(3\theta)]$$
$$z^n = r^n[\cos(n\theta) + i\sin(n\theta)]$$

$z = \sqrt{3} - i$

$z^3 = ?$

$r = \sqrt{3+1} = \sqrt{4} = 2$

$\tan\theta = -1/\sqrt{3}$

$\underline{\underline{\theta = -\pi/6}}$

$$\begin{aligned} z^3 &= 2^3[\cos(3(-\pi/6)) + i\sin(3(-\pi/6))] \\ &= 8[\cos(-\tfrac{3}{6}\pi) + i\sin(-\tfrac{3}{6}\pi)] \\ &= 8[\cos(-\pi/2) + i\sin(-\pi/2)] \\ &= \underline{\underline{-8i}} \end{aligned}$$

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Subsets of the complex plane.

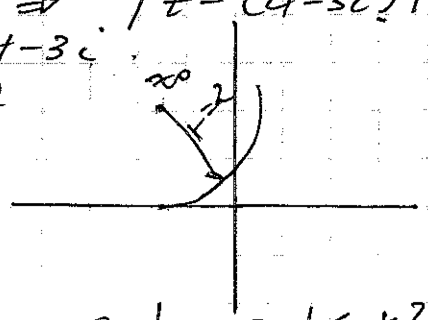
circle: $\{ z \in \mathbb{C} \mid |z - z_0| = r \}$

Graph $\{ z \in \mathbb{C} \mid |z - 4 + 3i| = 2 \}$

$\Rightarrow |z - (4 - 3i)| = 2$

$z_0 = 4 - 3i$

$r = 2$



Disc: $\{ z \in \mathbb{C} \mid |z - z_0| \le r \}$

open disk: $= \{ z \in \mathbb{C} \mid |z - z_0| < r \}$

↑ a nbhd of z_0

deleted nbhd of z_0

$\{ z \in \mathbb{C} \mid 0 < |z - z_0| < r \}$

Let $S \subseteq \mathbb{C}$

z_0 is an interior point of S if there exists a nbhd of z_0 entirely contained in S.

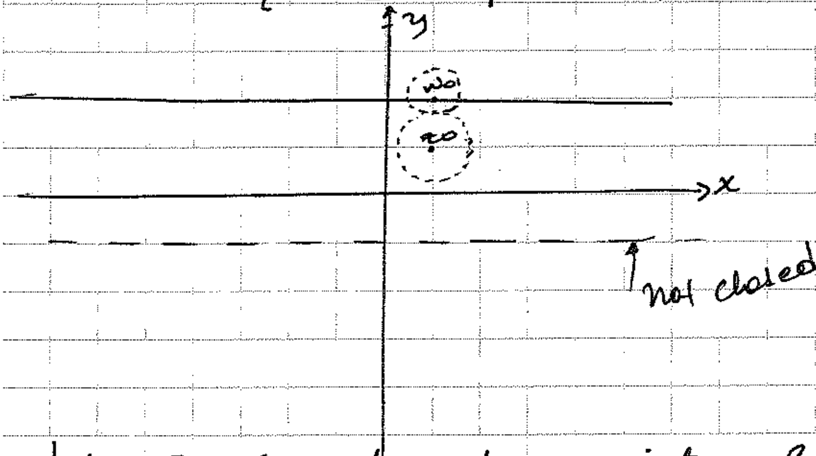
EX: $S = \{ z \in \mathbb{C} \mid -1 < \operatorname{Im}(z) \leq 2 \}$ let $z_0 = 1+i$

nbhd: $\{ z \in \mathbb{C} \mid |z-(1+i)| < 1/2$

$w_0 = 1+2i$

z_0 is an interior point

w_0 is not an interior point.



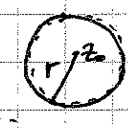
def^n z_0 is a boundary point of S if every nbhd about z_0 contain points in S & points outside S.

$w_0 = 1+2i$ is a boundary pt.

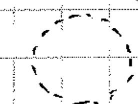
def^n - S is closed if it contains all its boundary units.

- If every point of S is an interior point, then S is open.

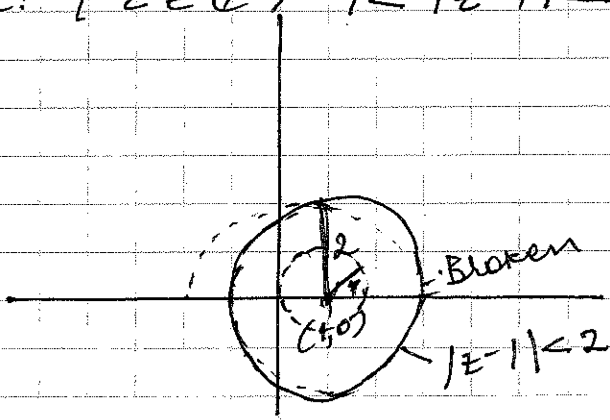
EX: A disk $\{ z \mid |z - z_0| \leq r \}$ is closed.



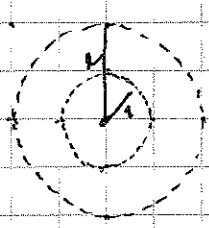
An open disk $\{ z \mid |z-z_0| < r \}$ is open



EX: $\{ z \in \mathbb{C} \mid 1 < |z-1| < 2 \}$



is S open? yes!



$1 < |z-1| < 2$

S is connected if any two points in S can be joined by a finite # of line segments within S.

An open connected set in $\mathbb{C}$ is called Domain.

S is bounded if there exists a real # R such that

$|z| < R$ for all $z \in S$

Sketch the follow, decide wheather it is bounded, closed open and even domain too.

i) $Re[(2+i)z+1]>0$

$Re[2z+iz+1]>0$

$Re[2(x+yi)+i(x+yi)+1]>0$

$Re[2x+2yi+xi-y+1]>0$

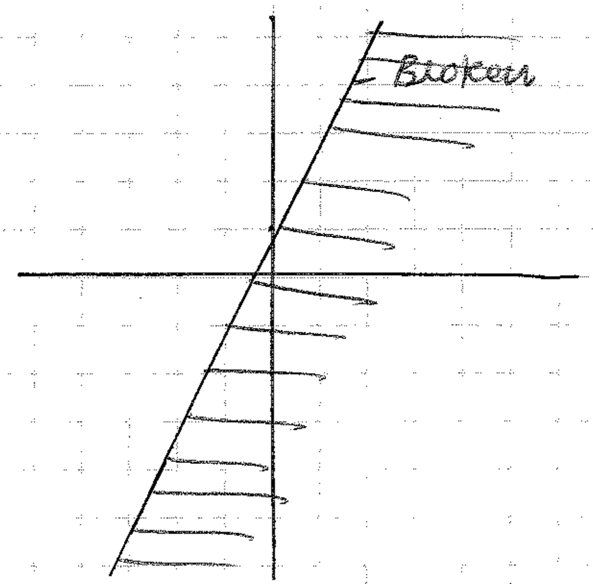
$Re[2x-y+1+(x+2y)i]>0$

$= 2x-y+1>0$

$\Rightarrow \{x+iy \,/\, 2x-y+1>0\}$

$-y>-1-2x$

$y<1+2x$



→ It is ~~not~~ open

→ It is not Bounded

→ It is connected

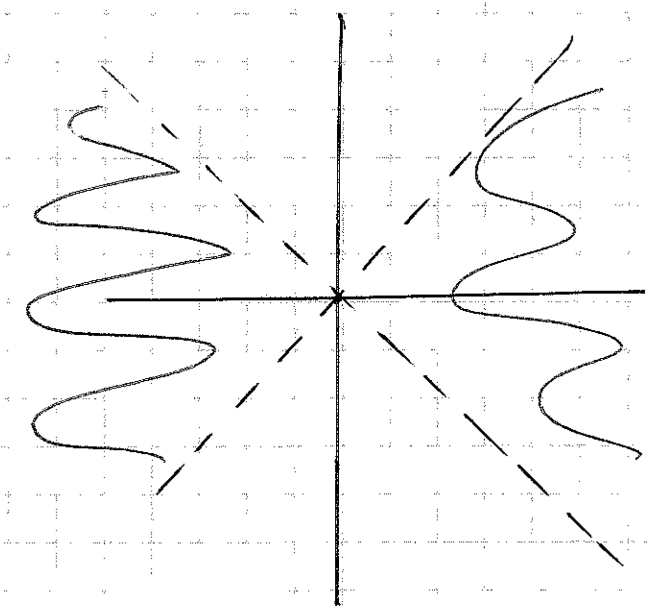
→ It is domain

ii) $Re(z^2)>0$

$Re(x+yi)^2>0$

$Re(x^2-y^2+2xyi)>0$

$x^2-y^2>0$



$-x \qquad x$

$(x-y)$ + + + + + + + 0 − − − −

$(x+y)$ − − − 0 + + + + + + + +

$(x-y)(x+y)$ − − 0 + + + + + 0 − − − −

— It is open

— It is not bounded

— It is not connected

— It is not a domain.

Quadratic Formula

$z^2+z+1=0$

$z=\dfrac{-1\pm\sqrt{1^2-4(1)(1)}}{2(1)}$

$=\dfrac{-1\pm\sqrt{3}i}{2}$

$=-\dfrac{1+\sqrt3}{2}i,\ -\dfrac{1-\sqrt3 i}{2}$

$z^2+iz-2=0$

$\dfrac{-i\pm\sqrt{i^2-4(1)(-2)}}{2(1)}$

$\dfrac{-i\pm\sqrt{7}}{2} \Rightarrow \dfrac{-i+\sqrt7}{2},\ \dfrac{-i-\sqrt7}{2}$

$z^2+2z-\sqrt3 i=0$

$\dfrac{-2\pm\sqrt{4-4(1)(-\sqrt3 i)}}{2}$

$\dfrac{-2\pm\sqrt{4+4\sqrt3 i}}{} = -1\pm\sqrt{1+\sqrt3 i}$

If $b^2-4ac<0$, $\sqrt{b^2-4ac}$ is complex $\therefore$ roots are complex conjugates!

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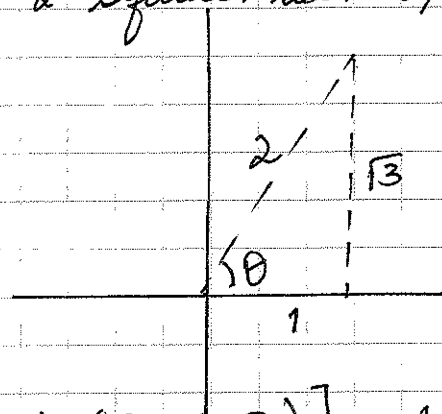
$$z = \frac{-b+(b^2-4ac)^{1/2}}{2a}$$ Find 2 square roots of b^2-4ac



$\sin\theta = \frac{\sqrt{3}}{2}$

$\cos\theta = \frac{1}{2}$

$\tan\theta = \sqrt{3}$

$\theta = \pi/3$

EX: $z^2+2z-\sqrt{3}i = 0$

$$z = \frac{-2+\sqrt{4-4(1)(-\sqrt{3})}}{2}$$

$$= -1+\sqrt{1+\sqrt{3}i}$$

if $\omega^2 = 1+\sqrt{3}i$

$$= 2[\cos(\pi/3)+i\sin(\pi/3)]$$

$$\omega = \sqrt{2}\left[\cos\left(\frac{\pi/3+2k\pi}{2}\right)+i\sin\left(\frac{\pi/3+2k\pi}{2}\right)\right], \quad k=0,1$$

$$\omega_0 = \sqrt{2}[\cos(\pi/6)+i\sin(\pi/6)] = \sqrt{2}(\sqrt{3}/2+i\,1/2)$$

$$\omega_1 = \sqrt{2}[\cos(7\pi/6)+i\sin(7\pi/6)] = \sqrt{2}(-\sqrt{3}/2+i(-1/2))$$

Substituting these values in

$$= -1+\sqrt{1+\sqrt{3}i}$$

$$z_0 = -1+\sqrt{2}\left(\sqrt{3}/2+\tfrac{1}{2}i\right)$$

$$z_1 = -1+\sqrt{2}\left(-\tfrac{\sqrt{3}}{2}-\tfrac{1}{2}i\right)$$

$$\omega^n = r[\cos\theta+i\sin\theta]$$

$$\boxed{\omega = r^{1/n}\left[\cos\left(\frac{\theta+2\pi k}{n}\right)+i\sin\left(\frac{\theta+2\pi k}{n}\right)\right]}, \quad k=0,1,2,\dots n-1$$

EX: $z^2+(1-i)z-3i=0$

$$z = \frac{-(1-i)+\sqrt{(1-i)^2-4(1)(-3i)}}{2}$$

$$= \frac{i-1+\sqrt{1^2-2i+i^2+12i}}{2}$$

$$= \frac{i-1+\sqrt{10i}}{2}$$

let $\omega^2 = 10i$

$$= 10(\cos(\pi/2)+i\sin(\pi/2))$$

$$\omega = \sqrt{10}\left[\cos\left(\frac{\pi/2+2\pi k}{2}\right)+i\sin\left(\frac{\pi/2+2\pi k}{2}\right)\right]$$

$$\omega_0 = \sqrt{10}[\cos(\pi/4)+i\sin(\pi/4)]$$

$$\omega_1 = \sqrt{10}[\cos(\tfrac{5}{4}\pi)+i\sin(\tfrac{5}{4}\pi)]$$

$$\omega_0 = \sqrt{10}\left[\tfrac{\sqrt{2}}{2}+i\tfrac{\sqrt{2}}{2}\right]$$

$$\omega_1 = \sqrt{10}\left[-\tfrac{\sqrt{2}}{2}+i\left(-\tfrac{\sqrt{2}}{2}\right)\right]$$

$$= \sqrt{10}\left[-\tfrac{\sqrt{2}}{2}-\tfrac{\sqrt{2}}{2}i\right]$$

$$z_0 = i-1+\sqrt{10}\left[+\tfrac{\sqrt{2}}{2}+i\tfrac{\sqrt{2}}{2}\right]/2$$

$$z_1 = i-1+\sqrt{10}\left[-\tfrac{\sqrt{2}}{2}-i\tfrac{\sqrt{2}}{2}\right]/2$$

Taylor Series

$$e^x = 1+x+x^2/2!+x^3/3!+x^4/4!+\dots+x^n/n!+\dots \quad R$$

$$\sin x = x-x^3/3!+x^5/5!-x^7/7!+\dots+ \quad R$$

$$\cos x = x+x^2/2!+x^4/4!-x^6/6!+\dots \quad R$$

For Complex number.

$$e^{i\theta} = 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} \cdots$$

collecting like terms

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right)$$

$$\boxed{e^{i\theta} = (\cos\theta + i\sin\theta)}$$

$\Rightarrow$ the exponetial form of a complex number is:

$$\boxed{re^{i\theta} = r(\cos(\theta) + i\sin(\theta))}$$

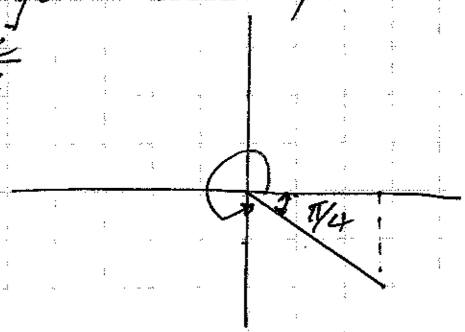
$$z^{1/n} = \sqrt[n]{r}\left[\cos\left(\frac{\theta+2\pi k}{n}\right) + i\sin\left(\frac{\theta+2\pi k}{n}\right)\right], \quad k = 0,1,2,\ldots,n-1$$

$$\boxed{z^{1/n} = \sqrt[n]{r}\, e^{i\left(\frac{\theta+2\pi k}{n}\right)}}$$

Ex: $z = 2/1+i$

write z in exp. form and find $z^{1/3}$

$$z = \frac{2}{1+i} * \frac{1-i}{1-i}$$
$$= \frac{2-2i}{1+1}$$
$$= \underline{1-i}$$



$$r = \sqrt{|1|^2} = \sqrt{2}$$

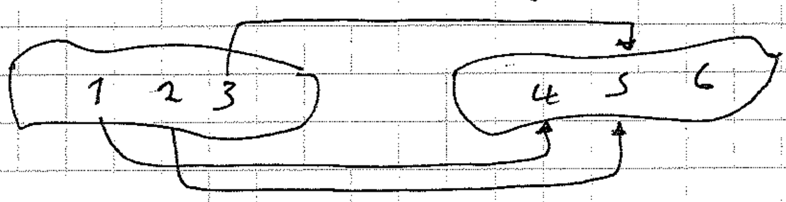
$$z = re^{i\theta} = r(\cos(\theta) + i\sin\theta)$$
$$= \sqrt{2}\, e^{i(-\pi/4)} = r\cos(-\pi/4) + i\sin(-\pi/4)$$
$$= \underline{\sqrt{2}\, e^{-\pi/4\, i}}$$

$$z^{1/3} = ?\ \sqrt[n]{r}\; e^{i\left(\frac{\theta+2\pi k}{n}\right)}$$
$$= \sqrt[3]{(2)^{1/2}}\; e^{i\left(\frac{-\pi/4+2\pi k}{3}\right)}, \quad k = 0,1,2$$

$$\begin{cases} z_0 = (2)^{1/6}\, e^{i(-\pi/12)} \\ z_1 = (2)^{1/6}\, e^{i(\frac{7}{12}\pi)} \\ z_2 = (2)^{1/6}\, e^{i(\frac{15}{12}\pi)} \end{cases}$$

Functions

A function from a set A to a set B is a rule which assigns to each element in A exactly one element in B.



$f(1)=4$
$f(2)=5$
$f(3)=5$

* if $b=f(a)$, then b is the image of a under f.

If $f: A \to B$, A = allowable input of f is called the domain of f.

the set of all output is called the range of f.

Ex: $\mathrm{dom}(f) = \{1,2,3\}$

Range $(f) = \{4,5\} \subseteq B$

Real functions: $f: \mathbb{R} \to \mathbb{R}$

Complex functions:

$f: \mathbb{C} \to \mathbb{C}$

If f is is a complex function the Input variable is $z = x+iy$

the Output variable is denoted by $w = u+iv$

ex: $f: \mathbb{C} \to \mathbb{C}$

Let $f(z) = z^2$

$$f(x+iy) = (x+iy)^2 = x^2-y^2+i2xy$$
$$= (x^2-y^2)+i(2xy)$$
$$= u(x,y)+iv(x,y)$$

Ex: 1. $f(z) = |z|$

2. $f(z) = z^{1/2}$ i) is f a function? why or why not?

3. $f(z) = z+\frac{1}{z}$ ii) if yes, find dom(f) & ran(f)

iii) if yes, write interms of Im & Re part

let ① $z = x+iy$

$\Rightarrow |z| = |x+iy|$

yes is a function

domain: all $\mathbb{C}$

Range: $[0, \infty]$

$u = \sqrt{x^2+y^2}$

$v = 0$

② $z^{1/2} = (x+iy)^{1/2}$

$z^{1/2} = re^{i\theta}$, $r = \sqrt{x^2+y^2}$

$= \sqrt{r}\, e^{i\left(\frac{\theta+2\pi k}{n}\right)}$ $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$= \sqrt[4]{x^2+y^2}\, e^{i\left(\tan^{-1}\left(\frac{y}{x}\right)+2\pi(0)\right)}$

$= (x^2+y^2)^{1/4} e^{i\tan^{-1}\left(\frac{y}{x}\right)/2}$

there are two output for each input $\Rightarrow$ not a function

③ $$(x+yi)+\frac{1}{(x+yi)} = \frac{x^2-y^2+2xyi+1}{(x+yi)} \cdot \frac{(x-yi)}{(x-yi)}$$

$$= \frac{x^3-xy^2+2x^2yi+x-x^2yi+y^3+2xy^2i^2-yi}{x^2+y^2}$$

$$= \underbrace{\frac{x^3+y^3+x-xy^2+2xy^2}{x^2+y^2}}_{u}+\underbrace{\frac{(2x^2y-x^2y-y)i}{x^2+y^2}}_{v}$$

domain = all $\mathbb{C}/0$

Range =

Exponential function:

What does $f(z) = e^z$ mean?

$$f(x+iy) = e^{x+iy}$$
$$= e^x e^{iy}$$
$$= e^x [\cos y + i \sin y]$$
$$= \underbrace{e^x \cos y}_{u = Re} + i \underbrace{e^x \sin y}_{v = Im}$$

If $f(z) = e^z$

Find $f(i)$ & $f(\pi i + 3)$

$z = i \Rightarrow x = 0,$ & $y = 1$

$f(i) = e^0 \cos(1) + i e^0 \sin 1$

$= \cos(1) + i \sin(1)$

$f(3 + \pi i)$

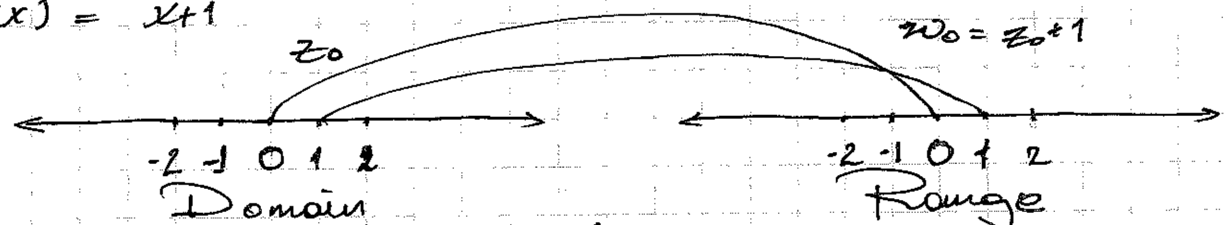
$x = 3$ & π

$f(3 + \pi i) = e^3 \cos \pi + i e^3 \sin \pi$

$= -e^3$

$f(x) = x + 1$

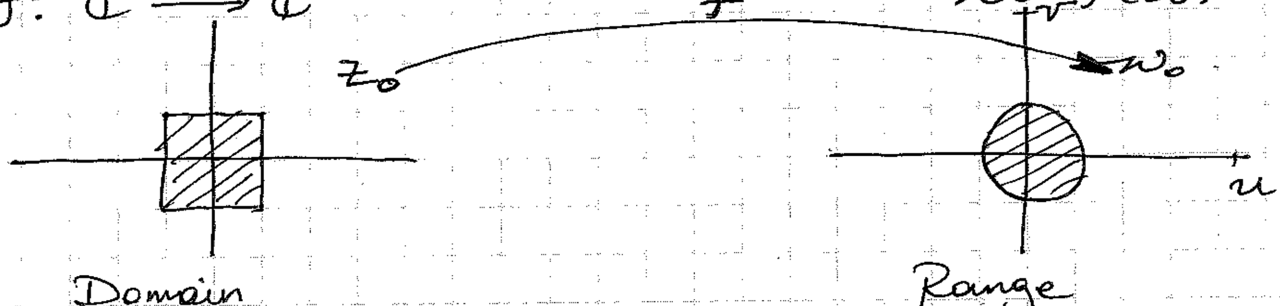
$w_0 = z_0 + 1$



Mapping

$f: \mathbb{C} \longrightarrow \mathbb{C}$

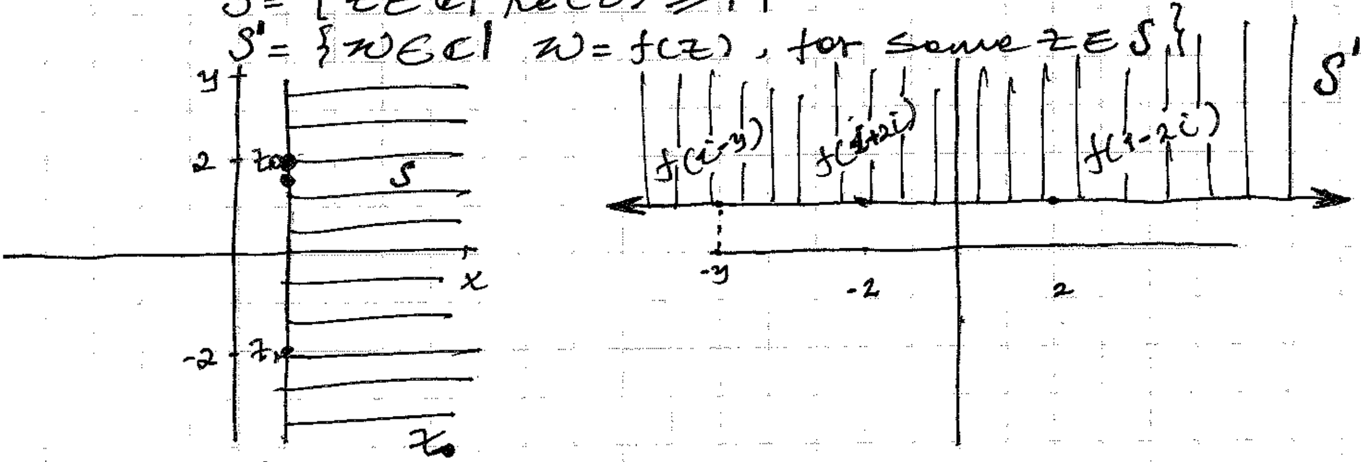
$w_0 = f(z_0)$



EX: $f(z) = iz$

$S = \{z \in \mathbb{C} \mid Re(z) \geq 1\}$

$S' = \{w \in \mathbb{C} \mid w = f(z), \text{ for some } z \in S\}$



$z_0 = 1 + 2i$

$f(z_0) = i(1+2i) = i - 2$

$z_1 = 1 - 2i$

$f(z_1) = i(1-2i) = i + 2$

$f(1+yi) = i(1+yi) = i - y$

if $-\infty < y < \infty$

$f(1+yi) = i - y$

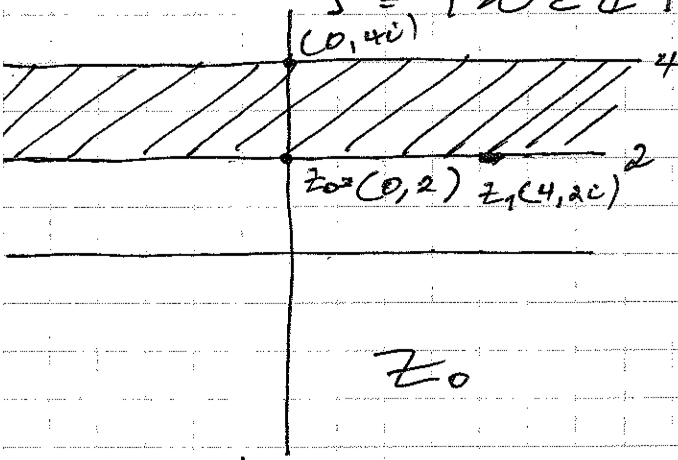
$= \{z \in \mathbb{C} \mid Im(z) = 1\}$

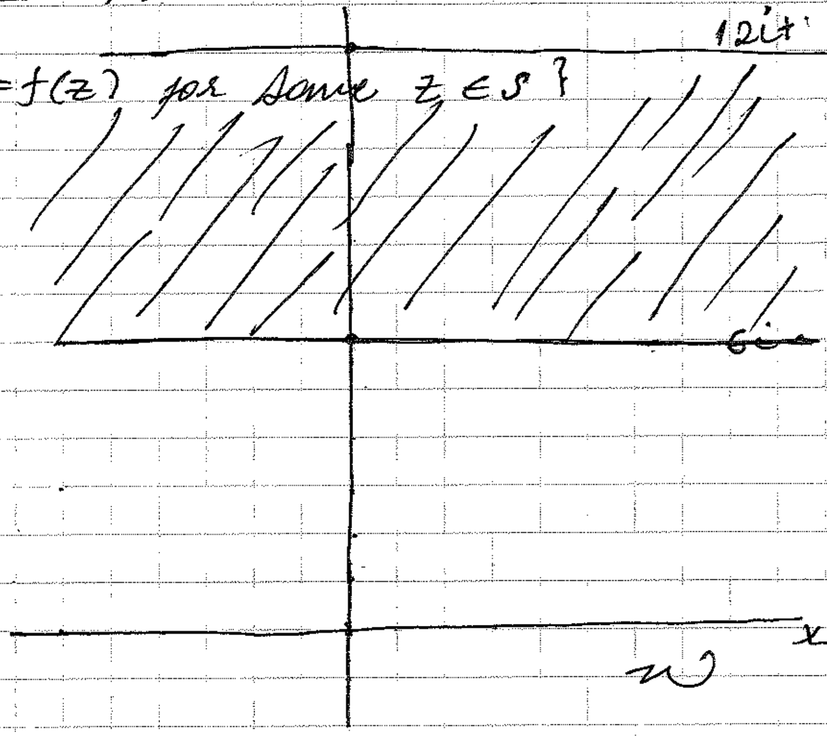
Suppose $z = x + iy$
where $x > 1$

$f(x+yi) = i(x+iy)$
$= -y + ix > -y+i$
$= \{ w \in \mathbb{C} \mid w < Re(w) < \infty \ \& \ Im(w) > 1 \}$

EX: $f(z) = 3z$
$S = \{ z \in \mathbb{C} \mid 2 \le Im \le 4 \}$.

Let the image S' of S be
$S' = \{ w \in \mathbb{C} \mid w = f(z) \text{ for some } z \in S \}$





$z_0 = 2i$
$f(z_0) = f(2i) = 3(2i) = \underline{6i}$
$f(z_1) = f(4,2i) = 3(4+2i) = 12+6i$
$f(z_2) = f(4i) = 3(4i) = 12i$

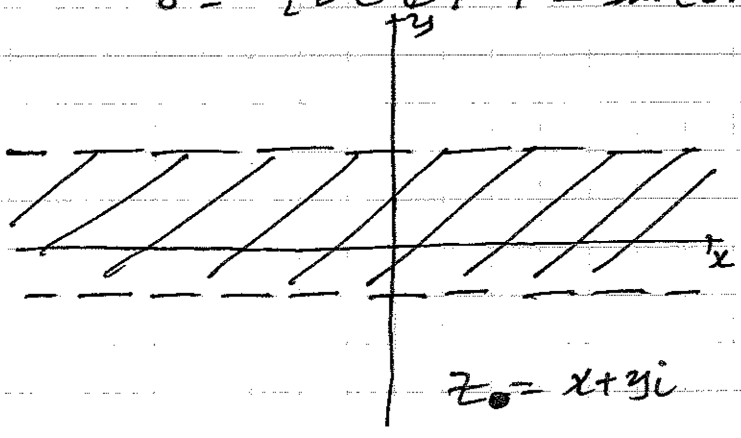
Restrictions
$-\infty < x < \infty$
$2 \le y \le 4$

$f(x+yi)$
$= 3x + 3yi$

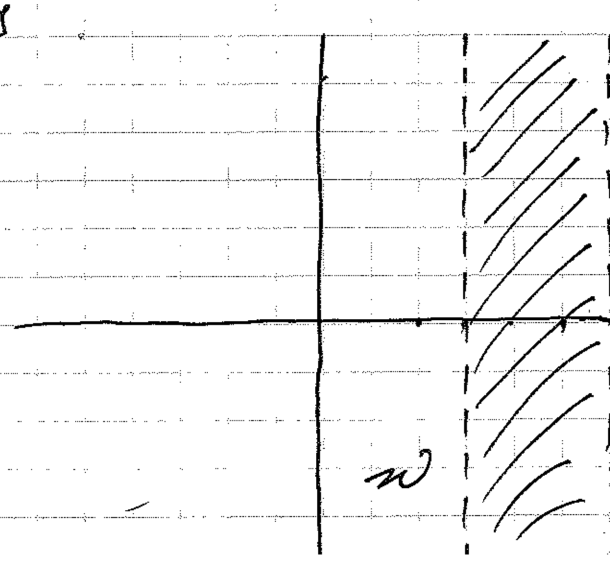
if y ranges $2 \le y \le 4$ then
$3y$ ranges $\underline{\underline{6 \le y \le 12}}$

H.W) $f(z) = z^2$
$S = \{ z \in \mathbb{C} \mid Im(z) = Re(z) + 1 \}$

EX: $f(z) = -iz + 4$
$S = \{ z \in \mathbb{C} \mid -1 < Im(z) < 2 \}$



$z_0 = x+yi$



$z = x + yi$

$f(z) = f(x+yi)$

$= -i(x+iy)+4$

$= -ix + y + 4$

$Re(z) = 4+y \quad u(x,y)$

$Im(z) = -x \quad v(x,y)$

$-1 < y < 2$

$-1+4 < y+4 < 2+4$

$3 < \underline{y+4} < 6$

$3 < Re(z) < 6$

The Homework.

$-\infty < x\ \infty$

$\infty < -x < -\infty$

$-\infty < Im(z) < \infty$

Ex: $f(z) = z^2$

$S = \{ z \in \mathbb{C} \mid Im(z) = Re(z)+1 \}$

S: $x = 3$

The line $x=3$ can be described as a set:

$S: \{ z \in \mathbb{C} \mid z = 3+iy,\ -\infty < y < \infty \}$

$f(z) = f(3+iy) = (3+iy)^2$

$= \underline{9-y^2} + \underline{6iy}$

$\left.\begin{array}{l} u(x,y) = 9-y^2 \\ v(x,y) = +6y \end{array}\right\}$ parametric equations

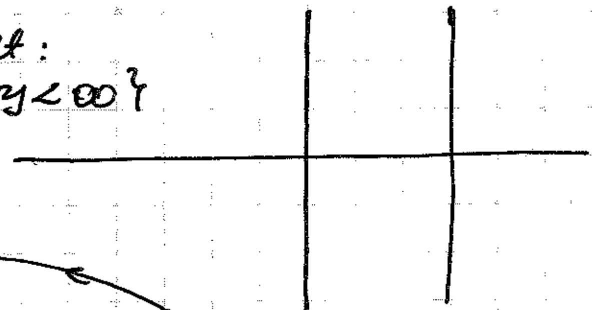
$y = v/6$

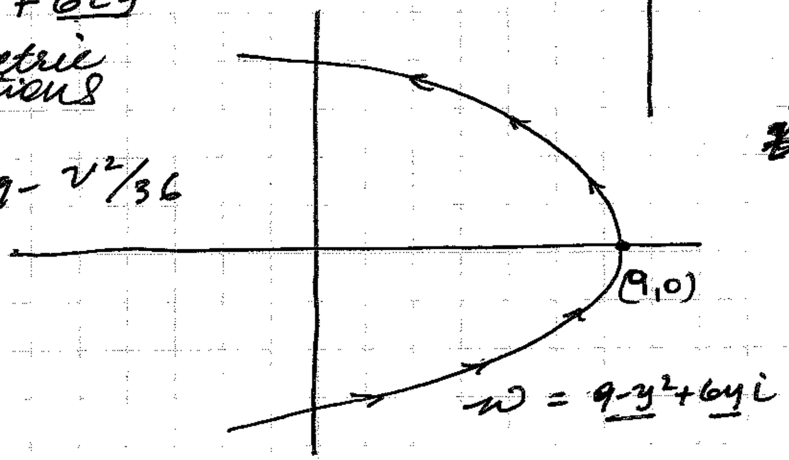
$u = 9 - (v/6)^2 = 9 - v^2/36$

$v = v/6$

$x =$ only 3

$-\infty < y < \infty$





More Generally,

If $x = x(t) \quad a \le t \le b$

$y = y(t)$

this traces at a curve in the complex plane:

$z(t) = x(t) + iy(t) \quad a \le t \le b$

called parametric equation for the curve.

Ex:

$z(t) = \underset{x(t)}{\cos t} + i \underset{y(t)}{\sin t}$

t	0	$\pi/2$	π	2π
(x,y)	$(1,0)$	$(0,1)$	$(-1,0)$	$(1,0)$

$0 \le t \le 2\pi$

$z(t) = \cos t + i \sin t$ → radius 1.

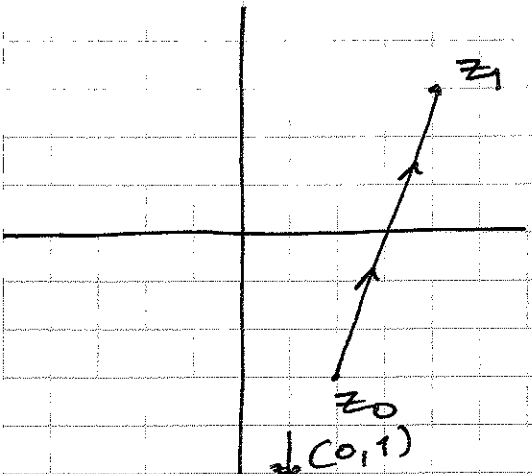
$= e^{it}$

$z(t) = r[\cos t + i \sin t] \quad 0 \le t \le 2\pi$

$= re^{it}$

circle centered at $[0,0]$ with radius r.

$z(t) = z_0 + re^{it}$ centered at z_0 radius r



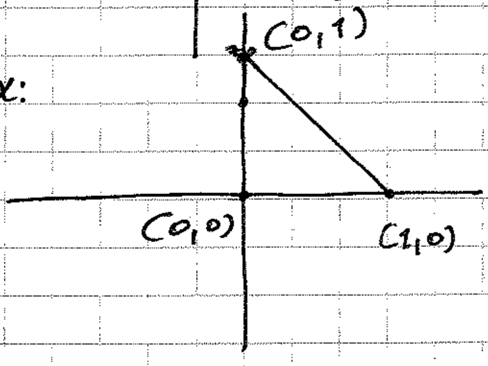
line segment between z_0 & z_1

$$z(t) = z_0(1-t) + z_1 t$$

$t=0 \ : z_0 \qquad\qquad 0 \le t \le 1$

$t=1 \ : z_1$

Ex:



Find line segment between

$z_0 = 0$

$z_1 = 1 + 0i$

$$\begin{aligned} z(t) &= z_0(1-t) + z_1 t \\ &= 0(1-t) + t \\ &= t, \quad 0 \le t \le 1 \end{aligned}$$

$z(t) = it, \quad 0 \le t \le 1$

$z_0 = 0 + 1i$

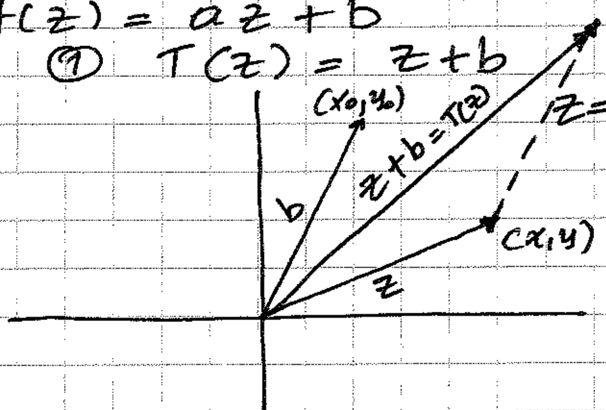
$z_1 = 1 + 0i$

$$\begin{aligned} z(t) &= (0+1i)(1-t) + 1(t) \\ &= t + i(1-t) \qquad 0 \le t \le 1 \end{aligned}$$

Linear Functions

$f(z) = az + b \qquad a, b \in \mathbb{C}$

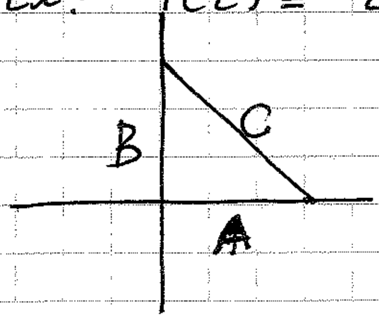
(1) $T(z) = z + b$

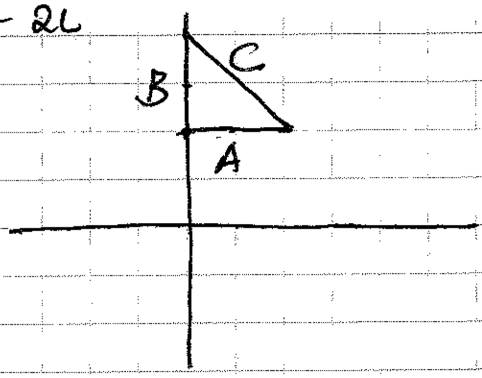


$z = (x, y) \qquad b = (x_0, y_0)$

(Translation of z by b)

Ex: $T(z) = z + 2i$





A: $z(t) = t, \quad 0 \le t \le 1$

$f(z(t)) = f(t) = T(t)$

$T(t) = t + 2i, \quad 0 \le t \le 1$

B: $z(t) = ti, \quad 0 \le t \le 1$

$$\begin{aligned} T(ti) &= ti + 2i \\ &= i(2+t) \end{aligned}$$

C: $z(t) = t + i(1-t) \quad 0 \le t \le 1$

$$\begin{aligned} T(t + i(1-t)) &= t + i - it + 2i \\ &= t + i(3-t) \end{aligned}$$

Ex: $R(z) = az$ where $|a| = 1$
is called Rotation. $-1 \leq a \leq 1$

$z = x + yi$

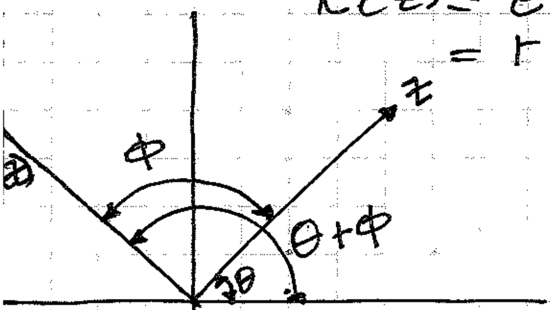
$z = r[\cos\theta + i\sin\theta]$

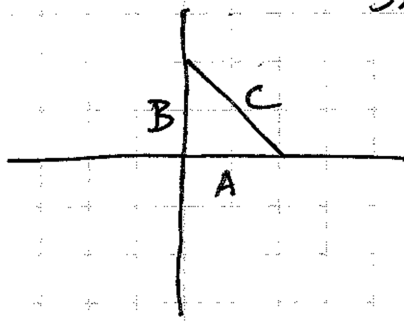
$z = re^{i\theta}$

$a = r e^{i\phi}$, $r \to 1$: $a = e^{i\phi}$

$R(z) = e^{i\phi} \cdot re^{i\theta}$
$= re^{i(\phi+\theta)}$

1) Translation
2) Rotation
3) Magnification





A: $z(t) = t$
B: $z(t) = it$
C: $z(t) = i(1-t)+1$

$0 \leq t \leq 1$

Ex: $R(z) = \left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)z$

A: $R(z) = \frac{\sqrt{2}}{2}$

$R(z) = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} = e^{i(\pi/4)}$, $z = re^{i\theta}$

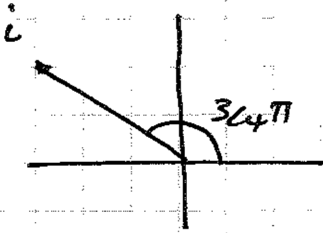
$R(z) = e^{i(\pi/4)} \cdot re^{i\theta} = re^{i(\theta+\pi/4)}$ $\quad \phi = \pi/4$

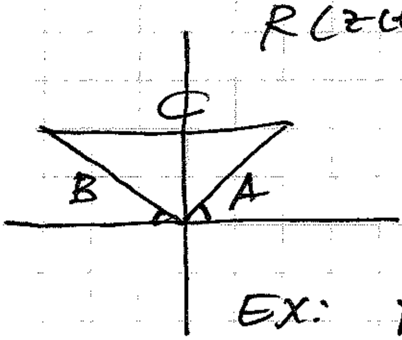
B: $z(t) = it$, $z(t) = it = te^{\frac{\pi}{2}i}$

$R(z(t)) = re$

$R(z(t)) = e^{i(\pi/4)} \cdot te^{\pi/2 i}$

$= te^{i(\pi/4+\pi/2)} = \underline{\underline{te^{i(3/4\pi)}}}$



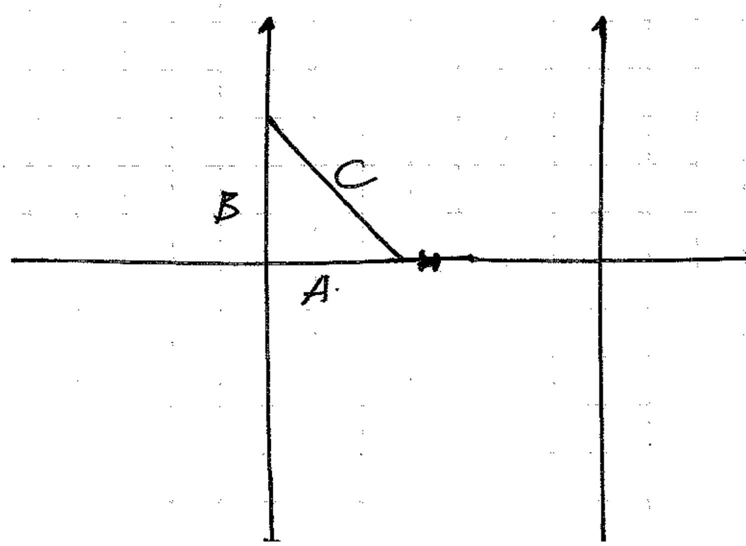


Magnification

Ex: $M(z) = 3z$ $\quad$ A $= M(z(t)) = 3t$, $0 \leq t \leq$

$F(z) = az + b$

$= \underbrace{|a|}_{\text{real scalar}} \underbrace{\left(\frac{a}{|a|}\right)}_{\text{on unit circle}} z + \underbrace{b}_{\text{translation}}$



Ex: $-\frac{1}{2}z + 1 - \sqrt{3}i = f(z)$

$f(z) = (+\frac{1}{2})(-z) + 1 - \sqrt{3}i$

(i) Translation $(1-\sqrt{3}i)$

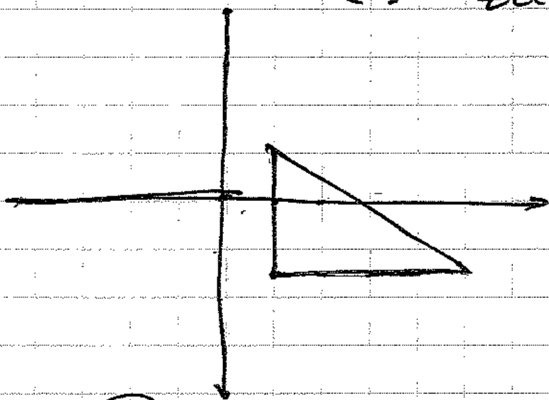
$z(t)$:

A: $z(t) = t$ $\quad f(z) = -\frac{1}{2}z + 1-\sqrt{3}i = t+1-\sqrt{3}i$

B: $z(t) = it$ $\quad f(z) = 1+(t-\sqrt{3})i$

C: $z(t) = i(1-t)+t$ $\quad f(z) = i - it + t + 1 - \sqrt{3}i$

$= t+1+i(1-t-\sqrt{3})$



Assume $f(z) = z + 1-\sqrt{3}i$

(ii) Rotation $\quad R(z) = -z \quad , \quad (-1) = e^{i\pi}$

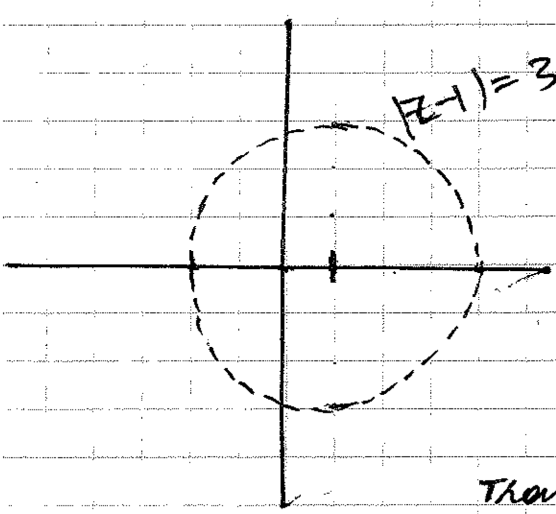
$z = re^{i\theta}$

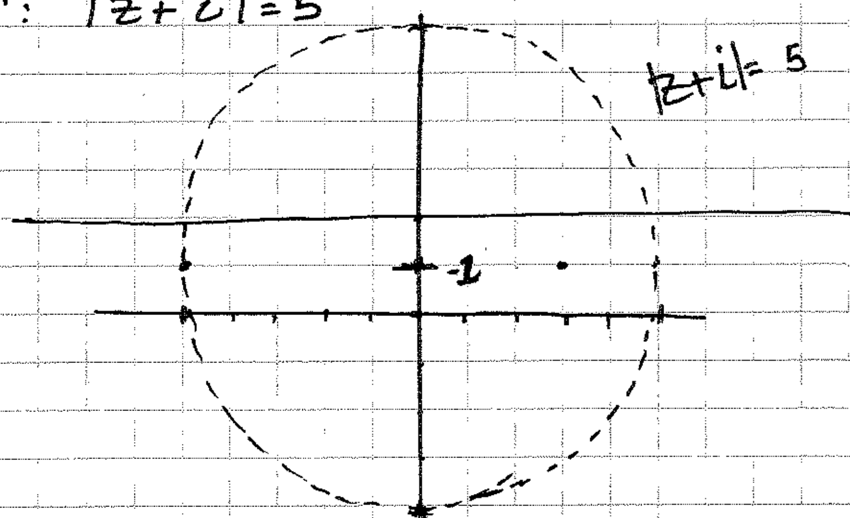
A: $z(t) = t, \quad R(z(t)) = e^{i\pi} \cdot t$

B: $z(t) = it, \quad R(z(t)) = e^{i\pi} \cdot it$

Ex: Find a linear map which maps the circle to

$S: |z-1| = 3 \quad$ to $\quad S': |z+i| = 5$





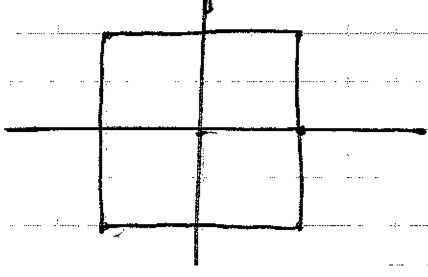
Translate by $1+i$

and multiply by $5/3$

$= f(z) = \frac{5}{3}z - 1 - i$

2) S: Square with vertices $1+i, -1+i, -1-i, 1-i$

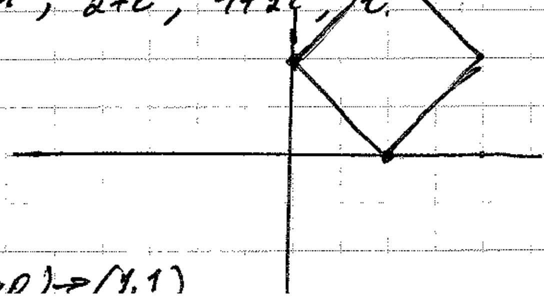
S': Square with vertices $1, 2+i, 1+2i, i$



Rotate by $\pi/4$

Shrink to $\sqrt{2}$

translate by $(0,0) \to (1,1)$



$R(z) = (\sqrt{2}/2 + i\sqrt{2}/2)\, z = e^{i(\pi/4)} \cdot z$. Rotate

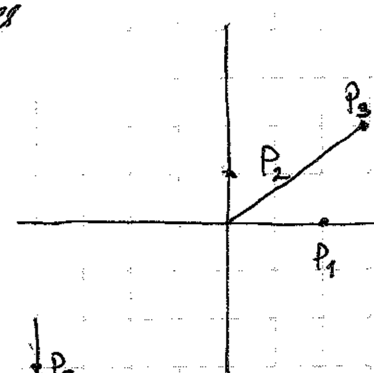
$M(z) = \frac{\sqrt{2}}{2}(\frac{\sqrt{2}}{2} + i\sqrt{2}/2)\, z = (1/2 + i\,1/2)\, z$ Shrink

$T(z) = z + i + 1$

$$f(z) = \frac{\sqrt{2}}{2}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) z + i + 1$$

Little more complecated function.

$$f(z) = \begin{cases} z^n \\ z^{1/n} \end{cases} \quad n \geq 2, \quad n \in \mathbb{Z} \to \text{Integers}$$



Warm up:

$f(z) = z^2$

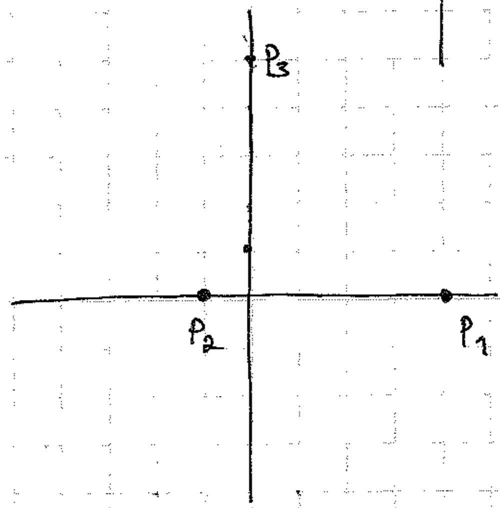
$f(3+4i) = (3+4i)^2 = -7 + 24i$

What about mapping

$z = r[\cos(\theta) + i\sin(\theta)]$

$= re^{i\theta}$

$f(z) = (re^{i\theta})^2 = r^2 e^{i(2\theta)}$



If $z = 2 \Rightarrow 2e^{i0}$

$f(2) = 4e^{i(0)} = \underline{4}$

$z = i \Rightarrow 1e^{i\pi/2}$

$f(i) = (e^{i\pi/2})^2 = \underline{-1}$

$z = 1+i$

$f(1+i) = \sqrt{2}e^{i\pi/4}$

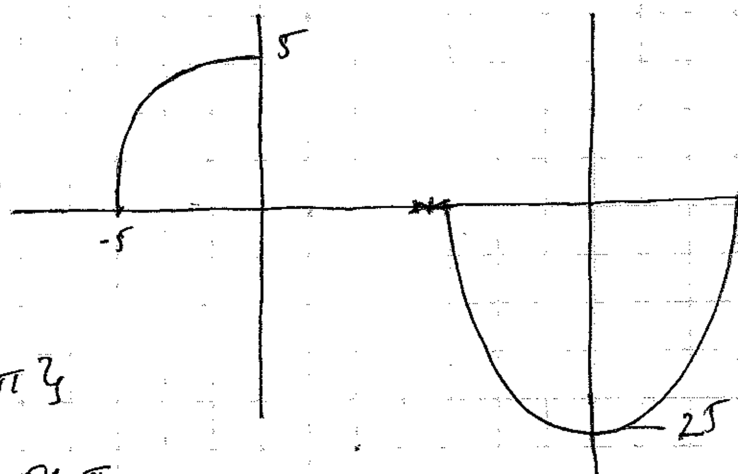
$f(\sqrt{2}e^{i\pi/4}) = (\sqrt{2}e^{i\pi/4})^2$

$= \underline{\underline{2i}}$

$f(z) \Rightarrow z = re^{i\theta}$

$= r^2 \cdot e^{i(2\theta)}$ → angle gets doubled. (rotation)

magnitude increase to r^2



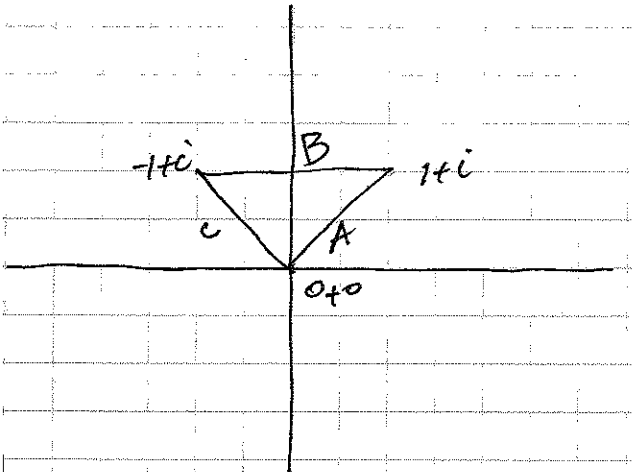
$S: \{ z \in \mathbb{C} \mid |z| = 5 \ \& \ \pi/2 \leq \theta \leq \pi \}$

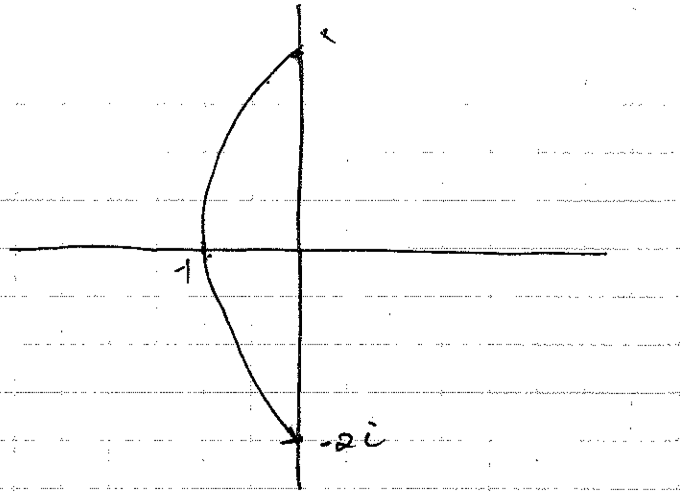
$S': \{ w \in \mathbb{C} \mid w = f(z), \ z \in S \}$

i.e. $z = re^{i\theta}, \ 5e^{i\theta}, \quad \pi/2 \leq \theta \leq \pi$

$z^2 = 25e^{i(2\theta)} \quad \pi/2 \leq \theta \leq \pi$

$\pi \leq 2\theta \leq 2\pi$





$$A = (0+0i)(1-t) + (1+1i)t$$
$$= t + it$$
$$B = (1+i)(1-t) + (-1+i)t$$
$$= 1 - t + i - ti - t + ti$$
$$= (1-2t) + i$$
$$C = (0,0)(1-t) + (-1+i)t$$
$$= -t + it$$

$$0 \le t \le 1$$

$S: \{z \in \mathbb{C} \mid$ triangle sirlin=

$S' = \{w \in \mathbb{C} \mid f(z) = z^2,\ z \in S$

$$A' = (t + it)^2$$
$$= t^2 - t^2 + 2t^2 i$$
$$B' = ((1-2t) + i)^2$$
$$= 1 - 4t^2$$

$$A = \sqrt{2}\, e^{i\pi/4} \qquad 0 \le r \le \sqrt{2}$$
$$B = \cancel{2}\, e^{i(0)} \qquad \cancel{0 \le r \le \sqrt{2}}$$
$$C = \sqrt{2}\, e^{i(3/4\pi)}$$

$$A' = f(z) = (\sqrt{2}\, e^{i\pi/4})^2$$
$$= 2e^{i\pi/2}, \quad 0 \le r^2 \le 2$$
$$= 2i$$
$$C' = \sqrt{2}\, e^{i(3/4\pi)} \qquad 0 \le r \le \sqrt{2}$$
$$= 2e^{i(\frac{3}{2}\pi)}$$
$$= -2i$$

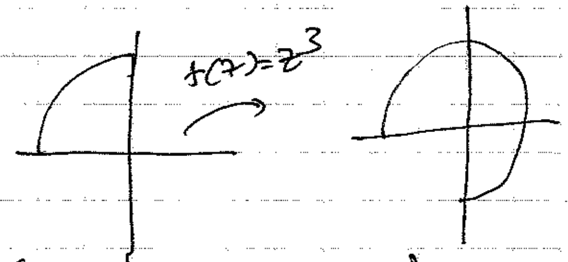
$$B = \quad y = 1$$
$$z = x + i, \quad -1 \le x \le 1$$
$$z^2 = \underbrace{x^2 - 1}_{u} + \underbrace{2x\, i}_{v} \qquad -1 \le x \le 1$$
$$u = x^2 - 1, \quad v = 2x$$
$$u = \frac{v^2}{4} - 1, \quad x = v/2$$
$$-2 \le v \le 2$$

$$= 2e^{i(\theta)} + i$$
$$= (2+i)^2 = 4 + 4i - 1$$
$$= 3 - 4i$$

Generally

$$f(z) = z^n \qquad n \ge 2$$
$$f(re^{i\theta}) = r^n e^{i(n\theta)}$$

$$z^3 = r^3 e^{i3\theta}$$
$$\tfrac{3}{2}\pi \le 3\theta \le 3\pi$$

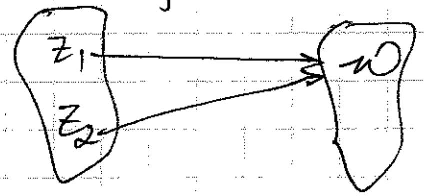


$S: \{z \in \mathbb{C} \mid |z| \le 5,\ \pi/2 \le \theta \le \pi\}$

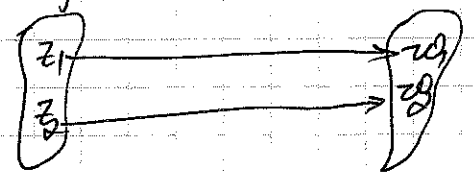
$S': \{w \in \mathbb{C} \mid f(z) = w\}$

Def^n. Let $f(z)$ be a complex function.
$f(z)$ is one-to-one if every element in the range of f is exactly the image of one element in the domain.

non-example



example



i.e. if $f(z_1) = f(z_2)$ then $z_1 = z_2$ $\qquad$ f is 1-1 means.

Prove: that $f(z) = z^2$ is not one to one.

$$f(-1) = f(1) = 1$$

Show that $f(z) = 2z + i$ is one-to-one.

S'pse $f(z_1) = f(z_2)$

$$2z_1 + i = 2z_2 + i$$
$$2z_1 = 2z_2$$
$$z_1 = z_2$$

$\therefore$ f is one-to-one

S'pse $f(z_1) = f(z_2)$

$z_1^2 = z_2^2$ or $z_1^2 - z_2^2 = 0$

$\underline{\underline{z_1 = \pm z_2}}$ $\qquad$ $(z_1 - z_2)(z_1 + z_2) = 0$

$\underline{\underline{z_1 = z_2}}$ or $\underline{\underline{z_1 = -z_2}}$

$$z_1 = z_2 = -z_2$$

$\therefore$ f is not 1-1

9/25/06

$f(z) = z^{1/n}$ $(n \geq 2)$

problem. $f(z) = z^{1/n}$ is not a function.

since if $z = re^{i\theta}$

then $z^{1/n} = \sqrt[n]{r}\, e^{i\left(\frac{\theta + 2\pi k}{n}\right)}$, $k = 0, 1, 2, \ldots, n-1$

Simplify:

① choose $k = 0$

② $\theta = \operatorname{Arg} z$ $(-\pi < \theta < \pi)$

define the principle n^{th} root function;

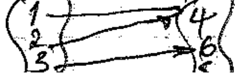
$f(z) = z^{1/n} = \sqrt[n]{r}\, e^{i\left(\frac{\operatorname{Arg} z}{n}\right)}$ → single-valued function.

If a function is multiple-valued, we use the notation.

$F(z) = z^{1/n}$

def^n $f(z)$ is one-to-one if no two elements in the domain get mapped to the same element under f.

non-example

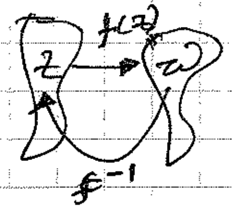


$f(1) = f(2) = 4$

$1 \neq 2$ not 1-1

FACT.

If f is one-to-one, then f is invertable.



Defn $f^{-1}(w) = f^{-1}(f(z)) = z$

$$f(f^{-1}(w)) = w$$

$$\boxed{f^{-1}(w) = z \quad \text{if } f(z) = w}$$

ex: $f(z) = iz + 1$

is $f(z) = 1\text{-}1$?

S'pse $f(z_1) = f(z_2)$

$f(z_1) = iz_1 + 1 = iz_2 + 1 = f(z_2)$

$iz_1 = iz_2$

$\underline{\underline{z_1 = z_2}}$

$w = iz + 1$

$z = (w-1)/i$

$z = -iw + i = f^{-1}(w)$

$w = iz + 1$

$z = iw + 1$

$= i(iz+1) + 1$

$= i - z + 1$

$f(z) = z^2$ not one-to-one since $f(i) = f(-i) = -1$

Q. Can we restrict the domain to make it 1-1.

A. Yes, we can,

Claim: $f(z) = z^2$ is 1-1 on $A: \{z \in \mathbb{C} \mid -\pi/2 < \arg z \le \pi/2\}$

why does this works.

S'pse $f(z_1) = f(z_2)$

$z_1 = r_1 e^{i\theta_1}$ $\quad f(z_1) = r_1^2 e^{i2\theta_1}$ $\quad r_1^2 e^{i2\theta_1} = r_2^2 e^{i2\theta_2}$

$z_2 = r_2 e^{i\theta_2}$ $\quad f(z_2) = r_2^2 e^{i2\theta_2}$

$\therefore r_1^2 = r_2^2$

$r_1 = r_2$ (since r_1, r_2 were positive)

$\mathrm{Arg}(r_1^2 e^{i2\theta_1}) = \mathrm{Arg}(r_2^2 e^{i2\theta_2})$

$\Rightarrow \cancel{\theta_1 = \theta_2}$

But θ_1, θ_2 $\quad -\pi/2 < \theta_1 \le \pi/2, \quad -\pi/2 < \theta_2 \le \pi/2$

$-\pi < 2\theta_1 \le \pi, \quad -\pi < 2\theta_2 \le \pi$

The principal argument $\mathrm{Arg}(r_1^2 e^{i2\theta_1}) = 2\theta_1$

$\mathrm{Arg}(r_2^2 e^{i2\theta_2}) = 2\theta_2$

$2\theta_1 = 2\theta_2$

$\theta_1 = \theta_2$

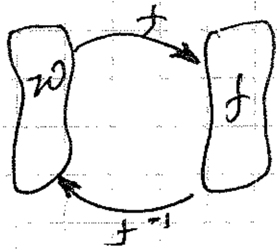
$f(z) = z^n$ is 1-1 on

$A: \{z \in \mathbb{C} \mid -\pi/n < \arg z \le \pi/n\}$

so $f(z) = z^2$ is invertable on $\{-\pi/2 < \arg z \le \pi/2\}$

what is its inverse

$f^{-1}(z) = z^{1/2} = \sqrt{r}\, e^{i \frac{\mathrm{Arg}\, z}{2}}$, where $z = re^{i\theta}$



Let $z = re^{i\theta}$ $\quad -\pi < \theta \le \pi$

$w = \rho e^{i\phi}$ $\quad -\pi < \phi \le \pi$

$w = f^{-1}(z)$

$\therefore w \in$ Range (f^{-1}) = domain$(f) = A$

$\therefore -\pi/2 < \phi \le \pi/2$

On the other H $\quad f(w) = w^2 = z$

So w is a square-root of z $\quad z = re^{i\theta}$

$$\therefore w = \sqrt{r}\, e^{i\theta/2} \quad \text{or} \quad w = \sqrt{r}\, e^{i(\frac{\theta+2\pi}{2})}$$

want to show that this is the principal argument value.

S'pse. $w = \sqrt{r}\, e^{i(\frac{\theta+2\pi}{2})}$

Since $-\pi < \theta \le \pi$

$$\frac{-\pi+2\pi}{2} \le \frac{\theta+2\pi}{2} \le \frac{\pi+2\pi}{2}$$

$$\frac{+\pi}{2} \le \frac{\theta+2\pi}{2} \le \frac{3\pi}{2}$$

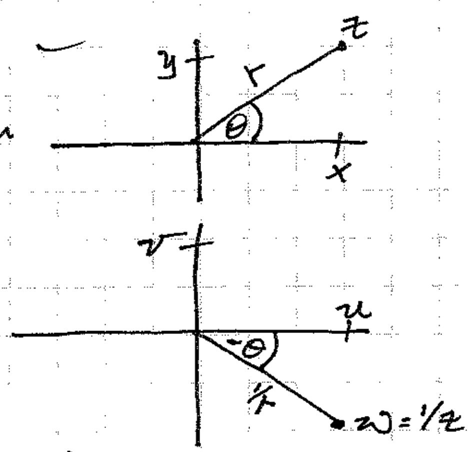
$$\frac{\pi}{2} < \phi \le 3\frac{\pi}{2}$$

2.5 Reciprocal Function

$f(z) = z^n \quad (n \ge 2)$

or $f(z) = 1/z^n \quad (n \ge 2)$

$n = 1$



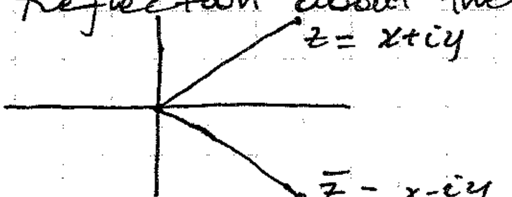
$f(z) = 1/z$

$z = re^{i\theta}$

if $w = f(z)$, then

$w = 1/z = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}$

Reflection about the x-axis.



$C(z) = \bar{z}$ = Conjugation

= reflection about x-axis.

$$C(z) = C(re^{i\theta}) = C(r\cos\theta + ir\sin\theta)$$
$$= r\cos\theta - ir\sin\theta$$
$$= r\cos(-\theta) + ir\sin(-\theta)$$
$$= re^{-i\theta}$$

Inversion on the unit circle.

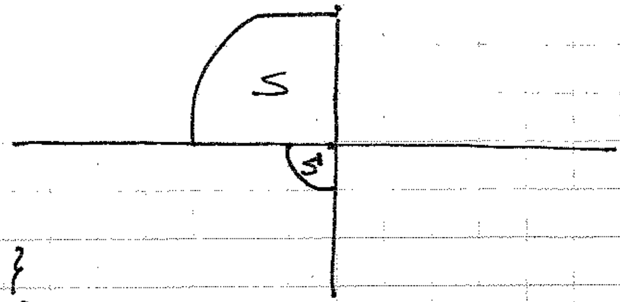
$g(z) = g(re^{i\theta}) = \frac{1}{r}e^{i\theta}$

if $r > 1$, then z is outside

$|z| = 1$,

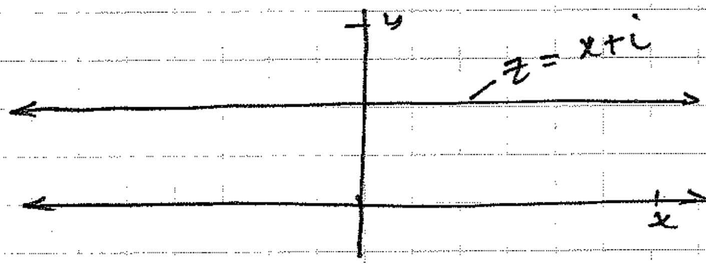
so $\frac{1}{r}e^{i\theta}$ is inside the unit circle.

$$f(z) = 1/z = \tfrac{1}{r} e^{-i\theta} = g(c(z))$$
$$= g(c(re^{i\theta}))$$
$$= g(re^{-i\theta})$$
$$= \tfrac{1}{r} e^{-i\theta}$$



$S = \{ z \in \mathbb{C} \mid |z| = 5 \quad \pi/2 \le \arg z \le \pi \}$
$S' = \{ w \in \mathbb{C} \mid |w| = 1/5 \quad \pi \le \arg w \le \tfrac{3}{2}\pi \}$

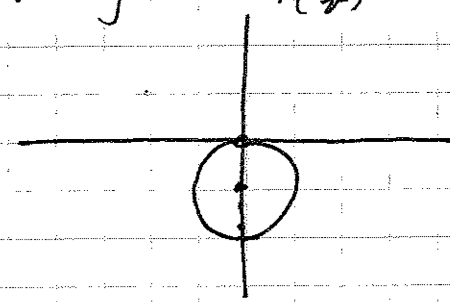
Q: what happens to lines?
Ex: $y=1$
$S: \{ z \in \mathbb{C} \mid z = x+i, \ -\infty < x < \infty \}$



$$f(z) = 1/z = \frac{1}{x+i} \cdot \frac{x-i}{x-i}$$
$$= \frac{x-i}{x^2+1} = \frac{x}{x^2+1} - i\frac{1}{x^2+1}$$

$u(x,y) = x/(x^2+1)$
$v(x,y) = -1/(x^2+1)$, $x^2+1 = -1/v$
$\boxed{u = \cancel{-x/v}} -xv$. $\rightarrow$ Substitute $v = -1/\left(\left(\frac{u}{v}\right)^2+1\right) = \frac{-v^2}{u^2+v^2} = v$
$x = -u/v$

$\Rightarrow -v = u^2+v^2$
$v^2 + v + u^2 = 0$
$u^2 + (v+\tfrac{1}{2})^2 = 1/4 = (1/2)^2$



The extended complex plane is the ordinary complex plane together with a "single point at ∞"
"point at ∞" - corresponds to points on the with arbitrary large modules.

Ex: $x = 3$
$z = iy + 3$
$$f(z) = 1/z = \frac{1}{3+yi} * \frac{3-yi}{3+yi}$$
$$= \frac{3-yi}{9+y^2}$$
$$= \frac{3}{9+y^2} - i\frac{y}{9+y^2}$$

$u = 3/(9+y^2)$
$v = y/(9+y^2)$, $9+y^2 = y/v$
$u = 3/(y/v) = 3v/y$
$u = 3v/y$
$\boxed{y = 3v/u}$

$v = y/(9+y^2)$, $u = 3/(9+y^2)$
$u = 3/(9+(3v/u)^2)$
$= 3/(9 + 9v^2/u^2)$
$u = 3u^2/(9u^2+9v^2)$
$9u^2 + 9v^2 = 3u$
$9v^2 + 9u^2 - 3u = 0$
$v^2 + u^2 - \tfrac{1}{3}u = 0$
$v^2 + (u - \tfrac{1}{6})^2 = 1/36 = (1/6)^2$
center at $(1/6, 0)$
$r = 1/6$
$S' = \{ w \in \mathbb{C} \mid |w - 1/6| = 1/6 \}$

FACT: Vertical

① The ~~Horizontal~~ line $x=k$ gets mapped to the circle $|z-\frac{1}{2k}|=\frac{1}{2k}$

② The Horizontal line $y=k$ gets mapped to the circle $|z+\frac{1}{2k}i|=\frac{1}{2k}$

< Under the map $f(z)=\frac{1}{z}$ >

Limit and Continuity

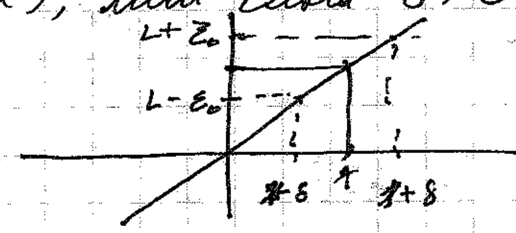
recall:

$$\lim_{x\to x_0} f(x) = L$$

Given $\varepsilon>0$ (no matter how small), there exists $\delta>0$

i.e. $0<|x-x_0|<\delta$

then $|f(x)-L|<\varepsilon$



Show $\lim_{x\to 1} 2x+1 = 3$

If $\epsilon>0$, δ i.e $0<|x-1|<\delta$ $\to$ $|x-1|<\delta$

$|2x+1-3|<\varepsilon$

$|2x-2|<\epsilon$

$2|x-1|<\varepsilon$

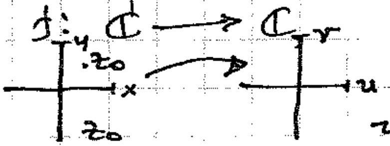
$|x-1|<\boxed{\varepsilon/2=\delta}>|x-1|$ going back $0<|x-1|<\varepsilon/2$

$0<|2x-2|<\varepsilon$

$|2x+1-3|<\varepsilon.$

For Complex:

$f: \mathbb{C}\to\mathbb{C}$



$$\lim_{z\to z_0} f(z) = L$$

$\lim_{z\to z_0} f(z) = L$ means: for $\forall\ \varepsilon>0$, there exist $\delta>0$ such that $0<|z-z_0|<\delta$, then $|f(z)-L|<\varepsilon$

$|f(z)-L|<\varepsilon$, is circle with radius ε and centered at L.

Proof:

find $\lim_{z\to 1-i} (2-i)z = 1-3i$

let $\epsilon>0$, $\exists$ some delta (δ) i.e $0<|z-(1-i)|<\delta$, then

$|f(z)-(1-3i)|<\varepsilon$

$|2z-iz-1-3i|<\varepsilon$

$|2z-1-i(z-3)|<\varepsilon$

$|2-i|\left|z-\frac{(1-3i)}{2-i}\right|<\varepsilon$

$|2-i|\left|z-\frac{5-5i}{5}\right|<\varepsilon$

$|2-i|\,|z-(1-i)|<\varepsilon$

$|z-(1-i)|<\varepsilon/\sqrt{5}$

$|z-(1-i)|<\delta \Rightarrow \underline{\varepsilon/\sqrt{5}=\delta}$

If $f(z)$ approaches L_1 along one path to z_0 & $f(z)$ approaches $L_2 \neq L_1$ along another path to z_0, then
$\lim_{z \to z_0} f(z)$ d.n.e.

Ex: $\lim_{z \to 0} \frac{z}{|z|}$ ① approach $z=0$ along x-axis.

$\lim_{z \to 0} z/|z|$, along x-axis $(0+0i)$

$= \frac{x+iy}{|x+iy|} \Rightarrow$

$$\lim_{(x+iy) \to (0+0i)} \frac{(x+y)i}{|x+yi|}$$

now along x-axis why $y=0$

$$= \lim_{x+i0 \to 0+0i} \frac{x+iy}{|x+iy|}$$

$$= \lim_{x \to 0} \frac{x+iy}{|x|}$$ from $+ = 1$, from $- = -1$

Ex:

Find the limit, $\lim_{x \to 3-2i} (1+i)z+1 = L$, proof too!

$L = (1+i)(3-2i)+1$

$L = 3-2i+3i-2i^2+1 = \underline{\underline{6+i}}$

$L = \underline{6+i}$

for $\forall \epsilon > 0$ there exist δ such that $0 < |z-(3-2i)| < \delta$, then

$$|f(z) - L| < \varepsilon$$
$$|f(z) - (6+i)| < \varepsilon$$
$$|(1+i)z+1-6-i| < \varepsilon$$
$$|(1+i)z-5-i| < \varepsilon$$
$$|(1+i)|\, \left|z - \left(\frac{5+i}{1+i}\right)\right| < \varepsilon$$
$$|1+i|\, \left|z - \left(\frac{6-4i}{2}\right)\right| < \varepsilon$$
$$|1+i|\, |z-(3-2i)| < \varepsilon$$
$$\sqrt{2}\, |z-(3-2i)| < \varepsilon$$
$$|z-(3-2i)| < \underline{\underline{\varepsilon/\sqrt{2} = \delta}}$$

$f(z) = f(x+iy) = u(x,y) + iv(x,y)$

$z_0 = x_0 + iy_0$

let $z \to z_0$

or $x+iy \longrightarrow x_0+iy_0$ or $(x,y) \longrightarrow (x_0,y_0)$

$\lim_{z \to z_0} f(z) = L$

iff $\lim_{(x,y) \to (x_0,y_0)} v(x,y) + iV(x_0,y_0)$

$$\lim_{z \to \pi i} e^z = \lim_{(x,y)\to(0,\pi)} e^{x+yi}$$
$$= \lim_{(x,y)\to(0,\pi)} e^x \cdot e^{yi}$$
$$= \lim_{(x,y)\to(0,\pi)} e^x[\cos y + i \sin y], \quad e^x \Rightarrow x=0 \Rightarrow e^x = 1$$
$$= \cancel{+}1$$

Continuity

Let $f(z)$ be a complex function.

$F(z)$ is continous at $z = z_0$ if,

$$\lim_{z \to z_0} f(z) = f(z_0)$$

must be defined   must be defined

EX:

$f(z) = z^3 - 1/z \quad z_0 = 3i$

is $f(z)$ c/s at $z_0 = 3i$

$f(3i) = (3i)^3 - 1/3i \qquad \lim_{z \to 3i} z^3 - 1/z = (3i)^3 - 1/3i$

EX: $f(z) = \dfrac{1}{|z|-1}$

c/s at $z = i$

Ans: no. because $f(i) = \dfrac{1}{|i|-1} = \dfrac{1}{0} \leftarrow$ undefined.

$$f(z) = \begin{cases} \dfrac{z^3-1}{z-1}, & |z| \neq 1 \\ 3, & |z| = 1 \end{cases}$$

c/s @ $z = i$?

$f(i) = 3$

$$\lim_{f(z)\to i} f(z) = \lim_{z \to i} \frac{z^3-1}{z-1} = \frac{(z-1)(z^2+z+1)}{(z-1)} = z^2+z+1$$

$$\lim_{z \to i} z^2+z+1 = -1+i+1$$
$$= \underline{\underline{i}}$$

not c/s.

Derivatives

S'pse $z = x+iy$

$z_0 = x_0 + iy_0$

want to compute the derivatives of $f(z)$ at $z \Rightarrow z_0$

Let $\Delta z = z - z_0$

$= (x+iy) - (x_0+iy_0)$
$= (x-x_0) + i(y-y_0)$
$= \Delta x + i\Delta y$

let $z \to z_0 \Rightarrow \Delta z \to 0$

dfn

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$$

find $f'(z)$ if $f(z) = z^2+z$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z_0+\Delta z)^2 + (z_0+\Delta z) - [z_0^2+z_0]}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\cancel{z_0^2} + 2z_0\Delta z + \Delta z^2 + \cancel{z_0} + \Delta z - \cancel{z_0^2} - \cancel{z_0}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{2z_0\Delta z + \Delta z^2 + \Delta z}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\cancel{\Delta z}(2z_0+\Delta z+1)}{\cancel{\Delta z}}$$

$$= \lim_{\Delta z \to 0} 2z_0 + \cancel{\Delta z} + 1$$

$$= \underline{\underline{2z_0+1}}$$

The power rule, sum, product, quotient, constant multiple rules, Chainrule.
all they holds and works.

$$f(x+iy) = x+3iy$$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{(\Delta x,\Delta y)\to(0,0)} \frac{f(x+iy+\Delta x+i\Delta y) - f(x+iy)}{\Delta x+i\Delta y}$$

$$= \lim_{(\Delta x,\Delta y)\to(0,0)} f((x+\Delta x)+i(y+\Delta y)) - f(x+iy) \,/\, \Delta x+i\Delta y$$

$$= \lim_{(\Delta x,\Delta y)\to 0} \cancel{x}+\cancel{\Delta x} + 3(y+\Delta y)i - \cancel{x} - \cancel{\Delta x} - (y-\Delta y)i \,/\, \Delta x+i\Delta y$$

$$= \lim_{(\Delta x,\Delta y)\to 0} (3y+3\Delta y - y - \Delta y)i \,/\, \Delta x+i\Delta y$$

$$= \lim_{\Delta x+i\Delta y \to 0} \Delta x - 3i\Delta y \,/\, \Delta x+i\Delta y$$

... be the same no matter path we

x-axis, $y=0$

$$\lim_{\Delta x \to 0} \frac{\Delta x - 3i\Delta y}{\Delta x + i\Delta y} = \lim_{\Delta x \to 0} \frac{\Delta x - 0}{\Delta x + 0} = \underline{1}$$

y-axis, $x=0$

$$\lim_{\Delta y \to 0} \frac{\Delta x - 3i\Delta y}{\Delta x + \cancel{3}i\Delta y} = \lim \frac{-3i\Delta y}{i\Delta y} = \underline{-3}$$

Differentiability

$f(z)$ is differentiable @ $z = z_0$ if

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z_0)}{\Delta z}$$

If so, the limit is the derivative at z_0: $f'(z_0)$

decide where $f(z) = \bar{z}$ is differentiable.

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$ let $z = x+yi$, $\bar{z} = \underline{x - yi}$

$$\lim_{\Delta x + i\Delta y \to 0} \frac{f(x+yi+\Delta x+i\Delta y) - f(x+yi)}{\Delta x + \Delta y i}$$

$$\lim_{\Delta x + i\Delta y \to 0} \frac{(\cancel{x}+\Delta x - (\cancel{y}+\Delta y)i) - \cancel{x}+yi}{\Delta x + \Delta y i}$$

$$\lim_{\Delta x - i\Delta y =} \frac{\Delta x - \Delta y i}{\Delta x + \Delta y i},$$ along x-axis $\lim_{x\to 0} f(x) = 1$

along y-axis $\lim_{y \to 0} f(x) = -1$

Def^n

$f(z)$ is analytic at z_0 if it is differentiable at every point in some nbhd of z_0

$f(z)$ is analytic on a domain if there is some domain for which $f(z)$ is analytic at every point in that domain.

Ex: $f(z) = z^2$

Find a domain for which $f(z)$ is analytic

$\mathbb{R} + \mathbb{C}$

A function which is analytic everywhere is called Entire.

What is partial derivative?

$z = f(x,y)$ is a function of x, y,

idea: hold one variable constant - get a function of one variable

ex: $f(x,y) = \sin(xy^2) + y^2x^2$

$f^*(x,2) = \sin(4x) + 4x^2$

$f'(x,2) = 4\cos 4x + 8x$

Theorem:

S'pse $f(z) = u(x,y) + iv(x,y)$ is differentiable at $z = x+iy$
Then at z - the first order partial derivatives of u & v exist.
the Cauchy-Riemann equation holds.

$$\frac{\partial u}{\partial x} = +\frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Proof:

hypothesis, $f'(z)$ exists:

$$\text{so } f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \quad \text{exists:}$$

$$= \lim_{\Delta x + \Delta y i \to 0} \frac{f(x+iy+\Delta x+\Delta y i) - f(x+iy)}{\Delta x + i\Delta y}$$

$$= \lim_{\Delta x + \Delta y i \to 0} \frac{f(x+\Delta x + i(y+\Delta y)) - f(x+iy)}{\Delta x + i\Delta y}$$

$$= \lim_{\Delta x + \Delta y i \to 0} \frac{u(x+\Delta x, i(y+\Delta y)) + iv(x+\Delta x, i(y+\Delta y)) - [u(x,y) + iv(x,y)]}{\Delta x + \Delta y i}$$

$$= \lim_{\Delta x + \Delta y i \to 0} \frac{u(x+\Delta x, i(y+\Delta y)) - u(x,y) + iv(x+\Delta x, i(y+\Delta y)) - iv(x,y)}{\Delta x + \Delta y i}$$

$$f'(z) = \lim_{\Delta x + \Delta y \to 0} \frac{u(x+\Delta x, y+\Delta y) - u(x,y)}{\Delta x + i\Delta y} + i\frac{[v(x+\Delta x, y+\Delta y) - v(x,y)]}{\Delta x + i\Delta y}$$

This limit exists $\Rightarrow$
the limit is independence of path.

① approach 0 along x-axis ($\Delta y = 0$) ($\Delta x \to 0$)

$$\lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i\frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$$

$$\boxed{f(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = f'(z)}$$

② approach 0 along y-axis ($\Delta x = 0$) ($\Delta y \to 0$)

$$\lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{\Delta y i} + \cancel{i}\frac{v(x, y+\Delta y) - v(x,y)}{\Delta y \cancel{i}}$$

$$\boxed{-i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = f'(z)}$$

equating ① & ②

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}}, \quad \boxed{\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$$

Gotta Corollary

If the Cauchy-Riemann equation donot hold at z
then f can not be differentiable @ z.

EX: $f(x) = \bar{z}$
$= x - yi$

$u(x,y) = x$ $\qquad \frac{\partial u}{\partial x} = 1 \quad \frac{\partial u}{\partial y} = 0$

$v(x,y) = -y$ $\qquad \frac{\partial v}{\partial x} = 0 \quad \frac{\partial v}{\partial y} = -1$

Caution:
Just because C-R equation holds, do not mean $f(z)$ is differentiable!

Criterion for Analyticity

S'pse u and v are continuous on a domain D & have continuous first partial derivatives on D. If u & v satisfy the C-R on D then f is analytic on D.

$f(z) = z^2$

$f(x+iy) = (x+iy)^2 = (x^2-y^2) + i(2xy)$

$u(x,y) = x^2 - y^2$ } Continuous everywhere $\qquad u_x = 2x \quad u_y = -2y$

$v(x,y) = \cancel{2}xy$ $\qquad v_x = 2y \quad v_y = 2x$

$$f(z) = \begin{cases} 0, & z = 0 \\ \frac{z^5}{|z^4|}, & z \neq 0 \end{cases}$$

$f'(0) = \lim_{\Delta z \to 0} \frac{f(0+\Delta z) - f(0)}{\Delta z}$
$= \lim_{\Delta z \to 0} \frac{f(\Delta z)}{\Delta z}$
$= \lim_{\Delta z \to} \frac{(\Delta z)^4}{|\Delta z^4|}$

$\Rightarrow \lim_{\Delta z \to 0^+} \frac{(\Delta z)^4}{\Delta z^4} = 1$
$\neq \Rightarrow$ not differentiab[le]
$\Rightarrow \lim \frac{(\Delta z)^4}{-\Delta z^4} = -1$

$f(z) = f(x+iy) = 3x^2y^2 - 6x^2y^2 i$

$u(x,y) = 3x^2y^2$ $\qquad u_x = 6xy^2$

$v(x,y) = -6x^2y^2$ $\qquad v_y = -12x^2y$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow u_x = v_y$

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow u_y = -v_x$

possible points of differentiability: are the xy-axes & that is it!

Thm

If $u(x,y)$ & $v(x,y)$ are continuous on a domain D, & u_x, u_y, v_x, v_y are continuous on D, and CR hold on the domain.

$f(z) = \frac{x-1}{(x-1)^2+y^2} - i\,\frac{y}{(1-x)^2+y^2}$

Find domain on which $f(z)$ is analytic.
avoid $(1,0)$

$u(x,y) = \frac{x-1}{(x-1)^2+y^2}$

$v(x,y) = \frac{-y}{(x-1)^2+y^2}$

$$u_x = \frac{(x-1)^2+y^2-(x+1)(2x-1)}{[(x-1)^2+y^2]^2}$$
$$= \frac{x^2-2x+1+y^2-[2x^2-3x+1]}{[(x-1)^2+y^2]^2}$$
$$= \frac{-(x-1)^2+y^2}{[(x-1)^2+y^2]} \quad //$$

$$v_y = -\frac{(x-1)^2-y^2+2y^2}{[(x-1)^2+y^2]^2}$$
$$= -\frac{(x-1)^2+y^2}{[(x-1)^2+y^2]^2} \quad //$$

$$u_y = \frac{-(x-1)(2y)}{[(x-1)^2+y^2]^2} \quad //$$

$$v_x = \frac{2(x-1)y}{[(x-1)^2+y^2]^2} \quad //$$

$\Rightarrow$ $\left.\begin{array}{l} u_x = v_y \\ \& \ u_y = -v_x \end{array}\right\} \Rightarrow$ The function is analytic all over the place as long as we avoid $(1,0)$.

what is the derivative?

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z}$$
$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$
$$\Rightarrow \frac{-(x-1)^2+y^2}{[(x-1)^2+y^2]^2} + i\,\cancel{\frac{-(x-1)^2+y^2}{}} + \frac{2(x-1)y}{[(x-1)^2+y^2]^2}$$
$$= \frac{-(x-1)^2+y^2-2y(x-1)i}{[(x-1)^2+y^2]^2}$$

Remark:

$$f(r,\theta) = u(r,\theta) + i\,v(r,\theta)$$

C-R equation:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

$$\boxed{f'(z) = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)}$$

Exponentials

$$e^z = e^{x+iy} = e^x e^{iy} = e^x[\cos y + i\sin y]$$
$$= e^x\cos y + ie^x\sin y$$

if $z = \text{real}$ $\quad \frac{d}{dx}(e^x) = e^x$

$e^z = e^x$

Fact: $\frac{d}{dz}(e^z) = e^z$, e^z is entire.

$u(x,y) = e^x\cos y \qquad u_x = e^x\cos y \qquad v_x = e^x\sin y$

$v(x,y) = e^x\sin y \qquad u_y = -e^x\sin y \qquad v_y = e^x\cos y$

$u_x = v_y$ ✓

$u_y = -v_x$ ✓ $\qquad f'(z) = e^x\cos y + ie^x\sin y = e^z$

$$= u_x + i v_x$$

ex: $f(z) = e^z z^3 - i z^2$, $f(z) = e^{z^2-(1+2i)z+10}$

$f'(z) = e^z z^3 + 3z^2 e^z - 2iz$ $\qquad f'(z) = e^{z^2-(1+2i)z+10}\cdot(2z-1-2i)$

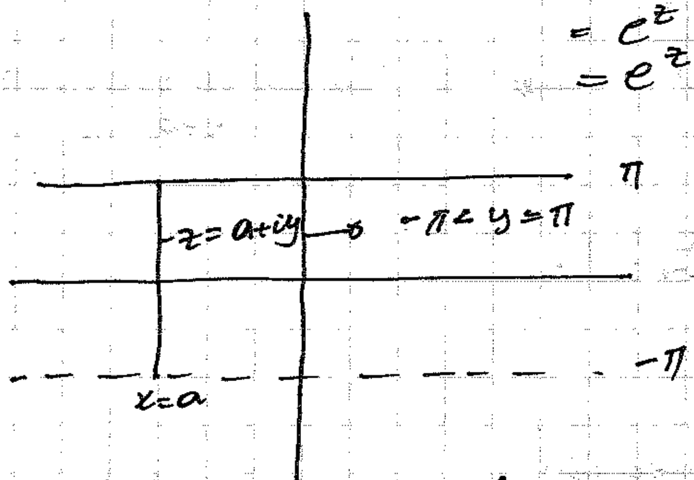
$f(z)$ is periodic with period T if $f(z+T) = f(z)$ $\forall z \in \mathbb{C}$

$f(x) = e^x$ is not periodic

But $f(z) = e^z$ is!

$T = 2\pi i$

$$f(z+2\pi i) = e^{z+2\pi i} = e^z \cdot e^{2\pi i} = e^z(\cos 2\pi + i\sin(2\pi)) = e^z = f(z)$$



$H = \{z \in \mathbb{C} \mid -\infty < x < \infty, -\pi < y \leq \pi\}$ is called the fundamental region for $f(z) = e^z$

What does $f(z) = e^z$ looks like:

$f(a+iy) = e^{a+iy} = e^a \cdot e^{iy}$ $\forall$ $-\pi < y \leq \pi$

e^z maps the fundamental region onto the punctured complex plane.

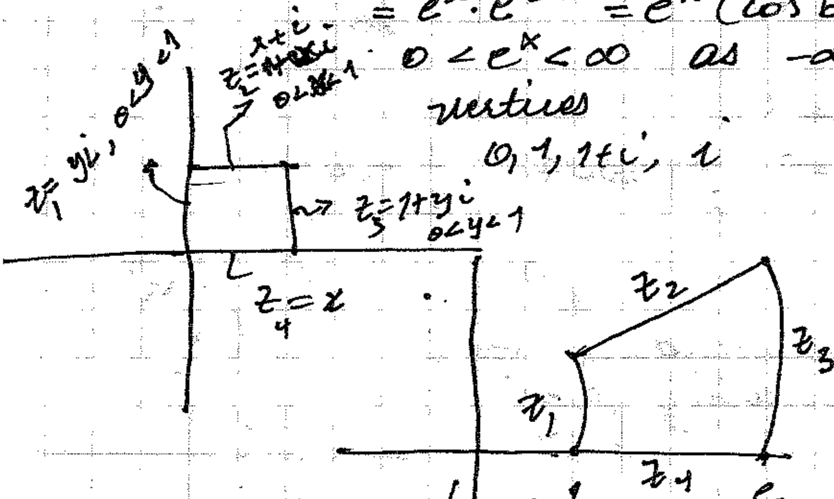
$f(x+ib) = e^{x+ib}$

$= e^x \cdot e^{ib} = e^x(\cos b + i\sin b)$

$0 < e^x < \infty$ as $-\infty < x < \infty$

vertices

$0, 1, 1+i, i$



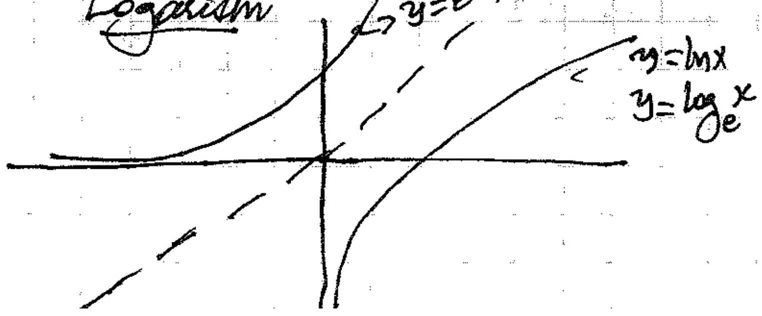
$e^{z_1} = e^{yi} = \cos y + i\sin y$

$e^{z_2} = e^{x+i} = e^x(\cos 1 + i\sin 1)$

$e^{z_3} = e^{1+yi} = e(\cos y + i\sin y)$

$e^{z_4} = e^{x+0i} = e^x(\cos 0 + i\sin 0)$

Logarithm



Want to define a complex log.

problem: e^z is not 1-1

Recall: if $e^y = x$

then $y = \log_e x$.

Want to solve $e^w = z$ (for z a fixed complex number)

Find w's which will work.

S'pse $w = u + vi$ satisfies $e^w = z$

If $e^w = z$, then

① $|e^w| = |z|$

② $\arg(e^w) = \arg(z)$

$$|e^w| = |e^{u+iv}| = |e^u \cdot e^{iv}| = |e^u| = e^u$$

So $|e^u| = |z|$

$$e^u = |z|$$
$$u = \log_e |z|$$

$$\arg(e^w) = \arg(z)$$
$$\arg(e^{u+iv}) = \arg(z)$$
$$v = \arg(z)$$

<u>Summarize</u>:

If $e^w = z$

then $w = \log_e(|z|) + i\arg(z)$. ↳ many values.

<u>def^n</u>

$\ln z = \log_e(|z|) + i\arg(z)$

"natural logarismic function"

multipled valued function!

$\operatorname{Ln} z = \log_e(|z|) + i\operatorname{Arg}(z)$

"Principal value of logarismic function"

$$\ln(1) = \log_e(|1|) + i\arg(1)$$
$$= \log_e(1) + i(0 + 2\pi n) \quad (n \in \mathbb{Z})$$
$$= 0 + 2n\pi i$$

$$\ln(-1) = \log_e(|-1|) + i\arg(-1)$$
$$= \log_e(1) + i(\pi + 2n\pi)$$
$$= i(2n+1)\pi \quad (n \in \mathbb{Z})$$

Solve:

$$e^w = (1 + \sqrt{3}i) = z$$

$$\ln(z) = \ln(1 + \sqrt{3}i)$$
$$= \log_e(|1 + \sqrt{3}i|) + i\arg(1 + \sqrt{3}i)$$
$$= \log_e 2 + i[\pi/3 + 2n\pi]$$
$$= \log_e 2 + \tfrac{1}{3}i[\pi + 6n\pi]$$

② $e^{2w-1} = 1$

$$e^2 e^w / e = 1$$
$$e^w = e/e^2$$

$e^{2w-1} = 1$

$\frac{e^{2w}}{e} = 1$

$e^{2w} = e$

$e^{w} = e^{1/2}$

$= \sqrt{e}$

$\ln z = \log_e(|\sqrt{e}|) + i \arg(\sqrt{e})$

$= \frac{1}{2}\log_e e + i \arg(\sqrt{e})$

$= \frac{1}{2} + 2n\pi i$

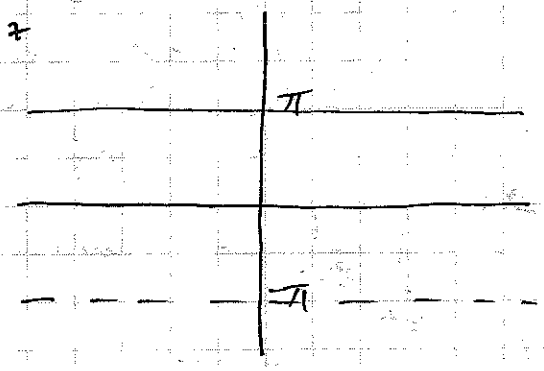
10/30/06

$e^{w} = z$

then $w = \ln z$ in particular $e^{\operatorname{Ln} z} = z$ suggests $\operatorname{Ln} z$ is the inverse of e^z.

But e^z is not 1-1 on its domain.

Restrict to the "fundamental region" of e^z



Claim:

$\operatorname{Ln} z$ is the inverse function of e^z on the f.r.

why? need to show $e^{\operatorname{Ln} z} = z \ \forall z \in$ F.R. $\operatorname{Ln}(e^z) = z$

Let $z = x + iy$ be in F.R.

so $-\infty < x < \infty$

$-\pi < y \le \pi$

$\operatorname{Ln} z = 10$

$\operatorname{Ln}(e^z) = \log|e^z| + i \operatorname{Arg}(e^z)$

$= \log_e e^x + iy$, since $z \in$ F.R. $\Rightarrow -\pi < y \le \pi$

$= x + iy$

$= z$

Multipled-valued functions:

e.g. $z = re^{i\theta}$

$z^{1/n} = \sqrt[n]{r}\, e^{i\left(\frac{\theta + 2k\pi}{n}\right)}$

be consistent w/ which n^{th} root we choose the principal n^{th} root of z is $\sqrt[n]{r}\, e^{i\frac{\operatorname{Arg} z}{n}}$ $-\pi < \operatorname{Arg} \le \pi$

Def^n A Branch of the multipled valued function $F(z)$ is a single valued function $f(z)$ that is continuous on some domain. (f assigns exactly one of the values of $F(z)$ on this domain

EX: One might guess that

$\sqrt{r}\, e^{i\frac{\operatorname{Arg}}{2}}$

is a branch of $z^{1/2}$

However, one can show that $\sqrt{r}\, e^{i\frac{\operatorname{Arg} z}{2}}$ is not continuous on the neg. real axis (i.e. $\theta = -\pi$)

Make it continuous by defining

$f_1(z) = \sqrt{r}\, e^{\frac{i\theta}{2}}$ $-\pi < \theta < \pi$

$\ln z = \log_e|z| + i \arg z$

to make this single valued function

$\operatorname{Ln} \stackrel{\text{def}}{=} \log_e|z| + i \operatorname{Arg} z$ $-\pi < \operatorname{Arg} \le \pi$

Q: is $\operatorname{Ln} z$ a branch of $\ln z$? Ans. No.

Claim:
$\operatorname{Ln} z$ is not continuous along the negative real axis.
1st: $\operatorname{Ln} z$ is not continuous at $z=0$ since it ain't defined there.
$\operatorname{Ln} z$ is continuous away from neg. real axis

$$\operatorname{Ln} z = \log_e |z| + i \operatorname{Arg} z$$
$$= \log_e \sqrt{x^2+y^2} + i \tan^{-1}\left(\frac{y}{x}\right)$$
$$= \frac{1}{2}\log_e (x^2+y^2) + .$$

↳ continuous from multivariable

$$z = re^{i\theta}$$

define $f_1(z) = \log_e(r) + i\theta$
$f_1(z)$ is a branch of $\ln z$
$$= \log_e |z| + i \operatorname{Arg} z$$
↳ almost

The negative real axis is a branch cut of $\ln z$ - a portion of the domain that is excluded to make the function cts. there

FACT: $f_1(z)$ is analytic on its domain &
$f_1'(z) = 1/z$

P

Recall: C-R eqns in polar coordinates:
$$f(r,\theta) = u(r,\theta) + i v(r,\theta)$$
C-R $\dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$ & $\dfrac{\partial u}{\partial \theta} = -\dfrac{1}{r}\dfrac{\partial v}{\partial r}$

$$f_1(r,\theta) = \underbrace{\log_e r}_{u} + i\underbrace{\theta}_{v}$$

$$\frac{1}{r} = \frac{1}{r}\cdot 1 \qquad 0 = -\frac{1}{r}\cdot 0$$

$u, v,$ & $\dfrac{\partial u}{\partial r}, \dfrac{\partial u}{\partial \theta}, \dfrac{\partial v}{\partial r},$ & $\dfrac{\partial v}{\partial \theta}$ all are continuous!

$$f'(r,\theta) = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$$
$$= e^{-i\theta}\left(1/r + i\cdot 0\right)$$
$$= \frac{e^{-i\theta}}{r} = \frac{1}{re^{i\theta}} = \frac{1}{z}$$

$e^{\ln z} = z \;\forall z$
$z^n = e^{\ln z^n} = e^{n\ln z}$
extend & define for any number α & $z \neq 0$
$$z^\alpha = e^{\alpha \ln z}$$
↳ return ∞'ly many values.

turns the same
what happens if α is an integer?
$$z^n = e^{n\ln z} = e^{n[\log_e|z| + i\operatorname{Arg} z]}$$
$$= e^{n[\log_e|z| + i(\operatorname{Arg} z + 2\pi k)]} \quad k \in \mathbb{Z}$$
$$= e^{n\log_e|z|}\, e^{ni\operatorname{Arg} z}\, e^{ni2k\pi}$$

$\Rightarrow e^{i2nk\pi} = \cos(2nk\pi) + i\sin(2nk\pi)$, since $n \& k \in \mathbb{Z}$

$= 1 + i0 = 1$

$\Rightarrow z^n = e^{n\log_e|z|} \cdot e^{ni\arg z}$

$\hookrightarrow$ single-valued.

In General $\boxed{z^n = e^{n\log_e|z|} \cdot e^{ni\arg z}}$

Ex: $i^i = e^{i\ln i} = e^{i[\log_e|i| + i\arg(i)]}$

let $z = i$, $n = i$

$\Rightarrow z^i$ where $z = i$

$$= e^{i[0 + i(\frac{\pi}{2} + 2k\pi)]}$$
$$= e^{i[\pi/2 + 2k\pi]}$$
$$= e^{(-\pi/2 - 2k\pi)} \quad k \in \mathbb{Z}$$
$$= \underline{\underline{e^{-\pi/2}}}$$

$(-1)^\pi \Rightarrow z = (-1)$, $n = \pi$

$z^n = (-1)^\pi$

$$z^n = e^{n[\log_e|z| + i\arg(z)]}$$
$$= e^{\pi[\log_e|-1| + i\arg(-1)]}$$
$$= e^{\pi[0 + i(\pi + 2\pi k)]}$$
$$= e^{\pi(i(\pi + 2\pi k))}$$
$$= e^{\pi^2 i + 2\pi^2 k i}$$
$$= \underline{\underline{e^{i\pi^2(1+2k)}}}$$

$$(-1)^{\frac{1}{\pi}} = e^{1/\pi \ln(-1)} = e^{i/\pi[\pi + 2\pi k]} = \underline{\underline{e^{i + 2ki}}}$$

If $z = re^{i\theta}$,

then $\ln z = \log_e r + i\theta \longleftarrow$ Principal branch of $\ln z$.

is single valued & analytic on $-\pi < \theta < \pi$

$\frac{d}{dz}(\ln z) = \frac{1}{z}$

$z^\alpha = e^{\alpha \ln z}$ is a single valued & analytic on the fun branch of the logarithm as well.

$\because$ if $-\pi < \theta \leq \pi$ $(z = re^{i\theta})$

then $\frac{d}{dz}(z^\alpha) = \frac{d}{dz}(e^{\alpha\ln z}) = \underbrace{e^{\alpha\ln z}}_{z^\alpha} \cdot \alpha \cdot \frac{1}{z} = z^\alpha \cdot \frac{\alpha}{z} = \underline{\underline{\alpha z^{\alpha-1}}}$

The principal value of z^α is $e^{\alpha \ln z} = e^{\alpha[\log_e|z| + i\arg z]}$

Calculate principal to $(-i)^i$

$z = (-i)$

$n = i$

$$z^n = (-i)^i = e^{i[\log_e|-i| + i\arg(-i)]}$$
$$= e^{i[0 + i\pi]}$$
$$= \underline{\underline{e^{-3\pi/2}}}$$

Compute $(-1+\sqrt{3}i)^{3/2}$

it is not $\pm\sqrt{8}$

$-\sqrt{8}$

Trigonometric functions:

If x is a real variable,

$$e^{ix} = \cos x + i \sin x$$
$$e^{-ix} = \cos(-x) - i\sin(x)$$
$$e^{ix} + e^{-ix} = 2\cos x$$
$$\Rightarrow \cos x = \frac{e^{-ix} + e^{ix}}{2}$$
$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} = \frac{e^{-ix} - e^{ix}}{2}$$

Define for z (a complex variable)

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$ define the other functions as expected. $$\tan z = \frac{\sin z}{\cos z}$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\frac{d}{dz}(\sin z) = \frac{d}{dz}\left(\frac{e^{iz} - e^{-iz}}{2i}\right) = \frac{ie^{iz} + ie^{-iz}}{2i} = \frac{e^{iz} + e^{-iz}}{2} = \cos z$$

Periodicity:

$$\sin(z+2\pi) = \frac{e^{i(z+2\pi)} - e^{-i(z+2\pi)}}{2} = \frac{e^{iz}\cdot \overset{1}{\cancel{e^{i2\pi}}} - e^{-iz}\cdot \overset{1}{\cancel{e^{-i2\pi}}}}{2}$$

$$= \frac{e^{iz} - e^{-iz}}{2} = \sin(z) \Rightarrow \text{It is periodic}$$

$$\sin z = i$$
$$e^{iz} - e^{-iz} = 2i$$
$$e^{iz} - e^{1/iz} = 2i$$ $\frac{1}{e^{i}}$

let $iz = t$

$$e^{t} - e^{-t} = 2i$$
$$e^{2t} - 1 = 2ie^{+t}$$
$$e^{2t} - 2ie^{t} - 1 = 0$$

let $e^{t} = y$ $\quad y^2 - 2iy - 1 = 0$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \quad a = 1, \; b = -2i, \; c = -1$$

$$\frac{2i \pm \sqrt{-4 - 4(1)(-1)}}{2(1)} = i$$

$$y = i = e^{t}$$
$$t = \log_e i = iz$$
$$z = -i\log_e i$$
$$= \log_e i^{1/i}$$
$$= \log_e i^{-i}$$

H.W

Relate $\tan hy$ to $\tan(iy)$

$\tan(iy)$

2.4	4.1
2.5	4.2
2.6	4.3
3.1	
3.2	

Solve:

$\sin z = -\cos z$ $\qquad$ let $e^{iz} = y$

$\dfrac{e^{iz}-e^{-iz}}{2i} = -\dfrac{e^{iz}+e^{-iz}}{2}$ $\qquad$ $y^2(1+i) = (1-i)$

$e^{-iz}-e^{iz} = ie^{iz}+ie^{-iz}$ $\qquad$ $(1+i)y^2 + (i+1) = 0$

$e^{-iz}-ie^{-iz} = ie^{iz}+e^{iz}$ $\qquad$ $a = (1+i)$

$e^{-iz}(1-i) = e^{iz}(1+i)$ $\qquad$ $b = 0$

$y^{-1}(1-i) = y(1+i)$ $\qquad$ $c = -(i+1)$

$(1-i) = y^2 + iy^2$

$\dfrac{\pm\sqrt{+4(1+i)(i+1)}}{2(1+i)} = \pm\dfrac{\sqrt{8i}}{2(1+i)}$

Inverse $\sin z = ?$

$\sin(z+2\pi) = \sin z \quad \forall z \Rightarrow$ not 1-1

$\therefore$ inverse $\sin z$ is an infinite valued function

solve

$\sin w = z$ - - - - - - - - - - - - (z fixed)

$\dfrac{e^{iw}-e^{-iw}}{2i} = z$

$e^{iw}-e^{-iw} = 2iz$

$e^{2iw}-1 = 2ize^{iw}$, let $e^{iw} = y$

$y^2-1 = 2izy$

$y^2-2izy-1 = 0$

$\dfrac{2iz\pm\sqrt{-4z^2-4(1)(-1)}}{2(1)} = \dfrac{2iz\pm\sqrt{-4z^2+4}}{2} = \dfrac{2iz\pm 2\sqrt{1-z^2}}{2} = iz\pm\sqrt{1-z^2}$

$y = e^{iw} = iz+\sqrt{1-z^2}$

$iw = \ln(iz+\sqrt{1-z^2})$

$\boxed{w = -i\ln(iz+\sqrt{1-z^2})}$

to define the principal value?

pick p.v of $(1-z^2)^{1/2}$ &

p.v of $\ln(\;)$:

EX: p.v $(\sin^{-1}\sqrt{2})$

$\sin^{-1}\sqrt{2} = w$

$\sin w = \sqrt{2}$

$w = -i\ln(i(\sqrt{2}) + (1-2)^{1/2})$

$= -i\ln(\sqrt{2}i + i)$

$= -i\ln(i(\sqrt{2}+1))$

$= -i\left[\log|i\sqrt{2}+1| + i\left(\frac{\pi}{2}+2\pi k\right)\right]$

$= -i\left[\log(\sqrt{2}+1) + i\frac{\pi}{2}\right]$

(1) Derivatives:

choose a branch of $w = \sin^{-1} z$.

$= -i\ln[iz+(1-z^2)^{1/2}]$

If we can choose a branch.

$w = \sin^{-1} z$

$z = \sin w$ differentiate with respect to z.

$\cos w \dfrac{dw}{dz} = 1$

$\dfrac{dw}{dz} = \dfrac{1}{\cos w} = \dfrac{1}{\cos(\sin z)}$

$= \dfrac{1}{(1-z^2)^{1/2}}$ //

$\sin^2 w + \cos^2 w = 1$

$\cos^2 w = 1 - \sin^2 w$

$\cos^2 w = (1-\sin^2 w)^{1/2}$

$= (1-z^2)^{1/2}$

FACT:

$\cos^{-1} z = -i \ln(z + i(1-z^2)^{1/2})$

$\tan^{-1} z = \dfrac{i}{2} \ln\left(\dfrac{i+z}{i-z}\right)$

$\cancel{\sin^{-1} z = 1/(1-z^2)^{1/2}}$

$\sin^{-1} z = -i \ln(iz + i(1-z^2)^{1/2})$

Derivatives:

$\dfrac{d}{dz}(\sin^{-1} z) = \dfrac{1}{(1-z^2)^{1/2}}$

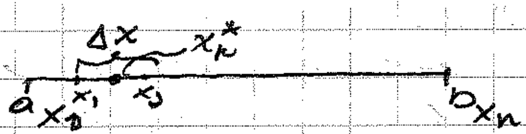
$\dfrac{d}{dz}(\cos^{-1} z) = \dfrac{-1}{(1-z^2)^{1/2}}$

$\dfrac{d}{dz}(\tan^{-1} z) = \dfrac{1}{1+z^2}$

Integration

Let $y = f(x)$ defined on $[a, b]$

$\int_a^b f(x)\,dx = ?$



Choose a partition

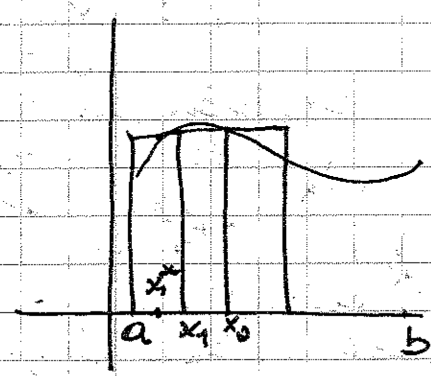
$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$

$\Delta x_k = x_k - x_{k-1}$

Choose $x_k^* \in [x_{k-1}, x_k]$

Form

$\sum_{k=1}^{n} f(x_k^*) \Delta x_k$



then let $\|P\|$ = norm of the partition = length of the largest subinterval

$\int_a^b f(x)\,dx = \lim_{\|P\| \to 0} \sum_{k=1}^{n} f(x_k^*) \Delta x_k$

Generalize

want to integrate over general curves:

Parametrically:

$x = x(t) \quad a \le t \le b$

$y = y(t)$

Circle: $x = \cos t$, $y = \sin t$

$0 \le t \le \pi$

let A & B be initial & terminal

A: initial pt: $(x(a), y(a))$

B: terminal pt: $(x(b), y(b))$