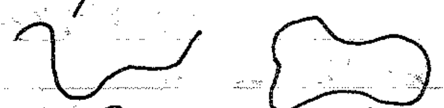
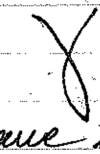
$A = [\cos(0), \sin(0)] = (1,0)$
$B = (\cos(\pi), \sin(\pi)] = (-1,0)$

Let C be a curve:

Def: C is smooth if $x'(t)$ & $y'(t)$ exists & are continuous through (a,b) & not simultaneously 0.

C is a piece wise smooth if it consists of finitely many smooth curves joined end to end

A curve is simple if doesn't cross itself except possibly at endpoints.

ex:  non-example: 

and a curve is closed if $A=B \Rightarrow$ both have the same initial and terminal points.

Line Integral

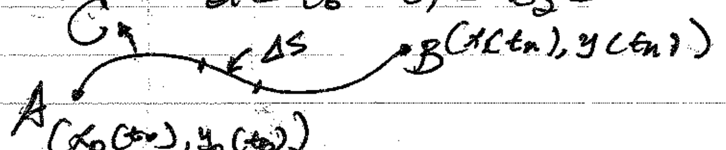
let $G(x,y)$ be a function of two variables.

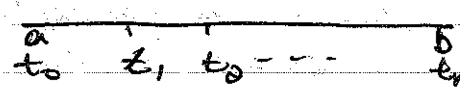
ex: $G(x,y) = x^2+y^2$

C: $x = x(t)$ $\quad a \le t \le b$
$\quad y = y(t)$

partition $[a,b]$ into n sub intervals.

$a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b$

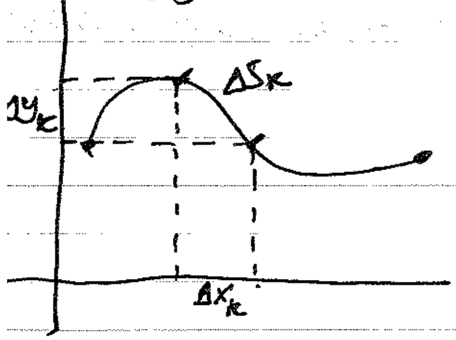




Let $\Delta S_k =$ length of the k^{th} subarc on C

$\Delta x_k \Rightarrow$ be the projection of ΔS_k onto x-axis.

$\Delta y_k \Rightarrow$ be the projection of ΔS_k onto y-axis.



Choose (x_k^*, y_k^*) in each ΔS_k

The line integral of G w.r.t. x is

$$\int_C G(x,y)\,dx = \lim_{\|P\| \to 0} \sum_{k=1}^{n} G(x_k^*, y_k^*)\,\Delta x_k$$

$$\int_C G(x,y)\,dy = \lim_{\|P\| \to 0} \sum_{k=1}^{n} G(x_k^*, y_k^*)\,\Delta y_k$$

$$\int_C G(x,y)\,ds = \lim_{\|P\| \to 0} \sum_{k=1}^{n} G(x_k^*, y_k^*)\,\Delta S_k$$

evaluate:

$\int_C x^2 y \, ds$, $x = 3\cos t$ $dx = -3\sin t$

$0 \le t \le \pi/2$ $y = 3\sin t$ $dy = 3\cos t$

$ds = \sqrt{dx^2 + dy^2} \cdot dt$

$= \sqrt{9\sin^2 t + 9\cos^2 t} \cdot dt$

$= 3\sqrt{\sin^2 + \cos^2 t}\, dt = 3\sqrt{1}\, dt = 3dt$

$\int 9\cos^2 t \,(3\sin t)\, 3dt$

$27 \int 3\cos^2 t \sin t \, dt$, let $\cos t = u$

$-27 \int 3u^2 \, du$ $du = -\sin t \, dt$

$-27 \left(\frac{3}{3} u^3\right)$ $0 \le t \le \pi/2$

$-27 u^3 \Big|_1^0$ $\cos 0 \le \cos t \le \cos \pi/2$

27 // $1 \le u \le 0$

Q: Change the curve but not the endpoints - will the value of the integral change?

Q: Can we change the parametrization & change the value. ans No the value remains the same.

— FACT:

The line integral, the value of a line integral over a curve is independent of the parametrization of the curve.

$y = t$

$x = 3 - t$ and opposite direction along the unit vector circle)

$0 \le t \le 3$

11/17/06

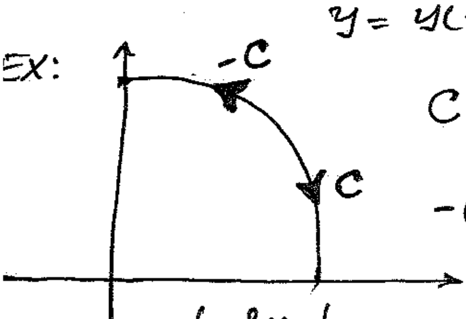
Note: About paths:

if C is a path, then $-C$ is the path traced in the opposite direction.

if C: $x = x(t)$ $a \le t \le b$ $-C$: $x = x(a+b-t)$ $a \le t$

$y = y(t)$ $y = y(a+b-t)$ $\le b$

EX:



C: $x = t$ $0 \le t \le 3$

$y = \sqrt{9 - t^2}$

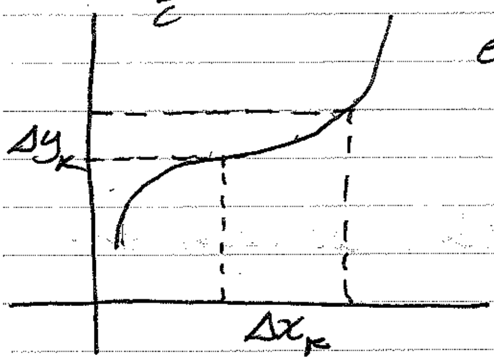
$-C$: $x = 3 - t$ $0 \le t \le 3$

$y = \sqrt{t^2 + 6t}$

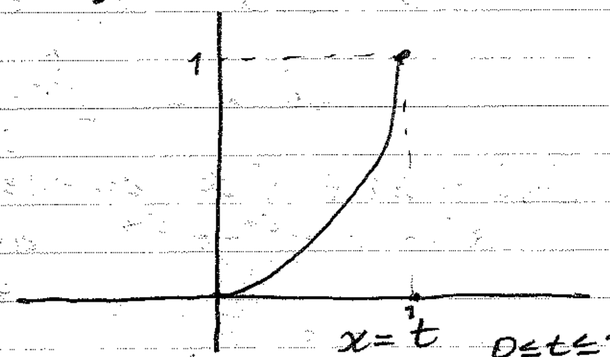
$\int_{-C} x^2 y \, ds = 27 \Rightarrow$ $\boxed{\int_C G(x,y)\, ds \text{, is independent of parametrization and also direction}}$

Typically, we are given two functions $P(x,y)$ & $Q(x,y)$.

$$\int_C P(x,y)\,dx + Q(x,y)\,dy = \int_C P\,dx + Q\,dy.$$



ex: $\int_C y\,dx + x\,dy$



$x = t$ $\quad 0 \le t \le 1$
$y = t^2$

$$\Rightarrow \int_C y\,dx + x\,dy$$

$$\int^1 t^2\,dt + t(2t)\,dt$$

$$\int_0^1 (t^2 + 2t^2)\,dt = \int_0^1 3t^2\,dt = \frac{3}{3}t^3\Big|_0^1 = \underline{\underline{1}}$$

$\Rightarrow$: $-C$: $\quad x = 1-t \qquad 0 \le t \le 1$
$\qquad\qquad y = (1-t)^2$

$$\int_{-C} y\,dx + x\,dx\,, \qquad \begin{aligned} dx &= -dt \\ dy &= 2(1-t)(-dt) = \cancel{2(t-1)}dt \end{aligned}$$

$$\int_0^1 -(1-t)^2\,dt + (1-t)(2(t-1))\,dt$$

$$-\int_0^1 \cancel{(1-t)^2\,dt} + \int 2(1-t^2 \quad -\int_0^1 t^2-2t+1\,dt - 2\int t^2-2t+1\,dt$$

$$-3\int t^2-2t+1\,dt = -3\left(\frac{1}{3}t^3 - \frac{2}{3}t^2 + t\right)\Big|_0^1$$

$$= -3(1/3 - 2/3 + 1)$$

$$= -\cancel{3}\,\frac{(1-2+3)}{\cancel{3}} = \underline{\underline{-1}}$$

Complex Line Integral.

Curve:

$$\left.\begin{aligned} x &= x(t) \\ y &= y(t) \end{aligned}\right\} \; z(t) = x(t) + i\,y(t) \qquad a \le t \le b$$

initial point: $z(a)$

terminal point: $z(b)$

C: is smooth if $z'(t)$ is continuous and never zero.

C: is simple if $z(t_1) \ne z(t_2)$ for $t_1 \ne t_2$ except at a & b

C: is closed if $z(a) = z(b)$

A piece wise - smooth curve in a complex plane is called a Contour.

want to define

$\int_C f(z)\,dz$.

$z(t) = x(t) + i\,y(t)$

$\cancel{\int_C f(z(t))\,d}$

$$\int_C f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt$$

Ex: $\oint_C \frac{1}{z}\,dz$ C: $x(t) = \cos t$, $y(t) = \sin t$ $0 \le t \le 2\pi$

$$= \int_0^{2\pi} \frac{1}{(\cos t + i\sin t)} \cdot (\cos t + i\sin t)'\,dt$$

$$= \int_0^{2\pi} \frac{\cos t - i\sin t}{\underbrace{\cos^2 t + \sin^2 t}_{1}} [-\sin t + i\cos t]\,dt$$

$$= \int_0^{2\pi} -\cancel{\cos t}\,\sin t + i\sin^2 t + i\cos^2 t + \cancel{\sin t\cos t}\,dt$$

$$= \int_0^{2\pi} (+i\sin^2 t + i\cos^2 t)\,dt$$

$$= i\int_0^{2\pi} (\cos^2 t + \sin^2 t)\,dt$$

$$= i\,[t]_0^{2\pi}$$

$$= \underline{\underline{2\pi i}}$$

EX: $\int_C \operatorname{im}(z-i)\,dz$

C: circle $|z|=1$ from $z = 1$ to $z = i$

$x(t) + iy(t) = \cos t + i\sin t$

$y = 0$ to $y = 1$

$|x+yi| = 1$ $\to 0 \le t \le \pi/2$

$\int \operatorname{im}(x+yi-i)\,dz$

$i\int (y-1)\,dz$

$i\int (\sin t - 1)(-\cos t + i\sin t)\,dt$

$i\int(-\sin t\cos t+\cos t+i\sin^2 t-i\sin t)\,dt$

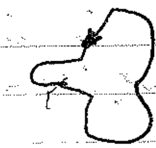
$\left[\int -i\sin t\cos t\,dt+\int i\cos t-\int \sin^2 t+\int \sin t\right]dt$

$-\frac{i}{2}(\sin t)\Big|_0^{\pi/2}+i(\cos\sin t)\Big|_0^{\pi/2}-\frac{1}{3}$

$= -(1+\frac{3}{2})+i(-\frac{1}{2})$

11/20/06

Contour Integrals.



C: curve in the complex plane.

$z(t)=x(t)+iy(t)\quad a\le t\le b$

$f(z)$ a complex valued function defined at all pts on C.

$$\int_C f(z)\,dz=\int_a^b f(z(t))\cdot z'(t)\,dt$$

Ex: $\int_C z^2\,dz$

C: $z(t)=3t+2it,\quad -2\le t\le 2$

$t=-2:\ z(-2)=-6-4i$

$t=2:\ z(2)=6+4i$

$x=3t,\quad y=2t$

$y=\frac{2}{3}x$

$\int_{-2}^{2} f(3t+2it)(3+2i)\,dt$

$\int_{-2}^{2}(3t+2it)^2(3+2i)\,dt$

$\int_{-2}^{2}(9t^2-4t^2+12ti)(3+2i)\,dt=\int_{-2}^{2}(5t^2+12ti)(3+2i)\,dt$

$\int_{-2}^{2}(15t^2+36ti+10t^2i-24t)\,dt$

$\int_{-2}^{2}(15t^2-24t)+i(36t+10t^2)\,dt$

$5t^3-12t^2+i(18t^2+\frac{10}{3}t^3)\Big|_{-2}^{2}$

$5(2)^3-12(2)^2+i(18(2)^2+\frac{10}{3}(2)^3)-[5(-2)^3-12(-2)^2+i(18(-2)^2+\frac{10}{3}(-2)^3)]$

$40-48+i(72+\frac{80}{3})+(+40)+48+i(\cancel{72}+\frac{80}{3})]$

$80-i(\frac{160}{3})$

EX: 2

$\int_C 1\,dz\quad$ C: left half of the ellips $\frac{1}{36}x^2+\frac{1}{4}y^2=1$

let $y=t\qquad -2\le t\le 2$

$x=\pm 6\sqrt{1-\frac{1}{4}t^2}\qquad \to x$ is always negative

$z(t)=x(t)+iy(t)$

$=6\sqrt{1-\frac{1}{4}t^2}+it$

$=\frac{6(-\frac{1}{2})t}{2\sqrt{1-\frac{1}{4}t^2}}-i$

$\Rightarrow\left(-\frac{3}{2}t\frac{1}{\sqrt{1-\frac{1}{4}t^2}}-i\right)dt$

$\Rightarrow\left[\int_{-2}^{2}\left(\frac{-3}{2}t\ast\frac{1}{\sqrt{1-\frac{1}{4}t^2}}\right)-\int_{-2}^{2} i\right]dt$

$$\text{let } 1-\tfrac{1}{4}t^2 = u, \quad du = -\tfrac{1}{2}t\,dt \Rightarrow -\frac{t\,dt}{2} = du$$

$$3\int_{-2}^{2}\left(\frac{1}{\sqrt{1-\frac{1}{4}t^2}}\,\frac{-t\,dt}{2}\right) - \int_{-2}^{2} i\,dt \qquad \frac{u^{-\frac{1}{2}+\frac{1}{2}}}{\frac{1}{2}}$$

$$6\int_{-2}^{2}\frac{1}{u^{1/2}}\,du - \left. 6\, it\right|_{-2}^{2}$$

$$\left. 6\,u^{1/2}\right|_{-2}^{2} - \left. 6 i\right|_{-2}^{2}$$

$$\left. 6\left(1-\tfrac{1}{4}t^2\right)\phi - 2i\,\right|_{-2}^{2}$$

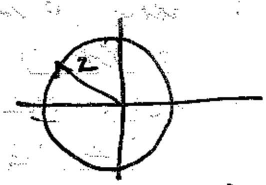
$$\left[6\left(1-\tfrac{1}{4}(2)^2\right) - 2i\right] - \left[6\left(1-\tfrac{1}{4}(-2)^2\right) - (-2)i\right]$$

$$6(0) - 2i - 6(0) - 2i$$

$$\underline{\underline{4i}}$$

Ex:

$$\int_{|z|=2}\frac{1}{z^2-1}\,dz \quad \text{Counter clock wise}$$



$$z(t) = x(t) + i\,y(t)$$

$$x = \cos t$$

$$y = \sin t \qquad -2 \le x \le 2, \quad -2 \le y \le 2$$

$$\int_{|z|=2}\frac{1}{(x+yi)^2-1}$$

$$\int_{|z|=2}\frac{1}{\cos^2 t - \sin^2 t + 2\cos t\sin t\, i - 1} = \int_{|z|=2}\frac{1}{2\cos^2 t - 2 + 2\cos t\sin t}\,dz$$

$$\int_{|z|=2}\frac{1}{-2\underbrace{(1-\cos^2 t)}_{\sin^2 t} + 2\cos t\sin t}\,dz = \int_{|z|=2}\frac{1}{-2\sin t\,(\sin t - \cos t)}\,dz$$

$$A = -B$$

$$A - B = 2$$

$$-2B = 2$$

$$B = -1$$

$$A = 1$$

$$z(t) = \cos t + i\sin t$$

$$dz =$$

$$|z| = 2 \Rightarrow z(t) = 2e^{it} \quad 0 \le t \le 2\pi$$

$$z'(t) = 2ie^{it}$$

$$\int_{|z|=2}\frac{1}{z^2-1}\,dz = \int_{0}^{2\pi}\frac{1}{(2ie^{it})^2-1}\,2ie^{it}\,dt$$

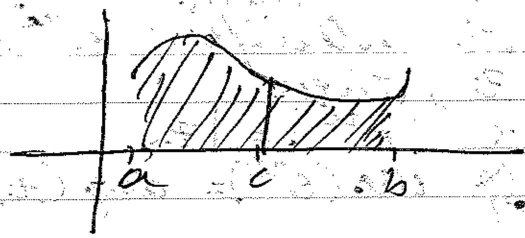
$$= \int_{0}^{2\pi}\frac{1}{-4e^{2it}-1}\,2ie^{it}\,dt, \quad \text{let } -1-4e^{2it} \text{ not}$$

$$\text{let } u = e^{it}$$

$$du = ie^{it}\,dt$$

Calc I.

$\int_a^b f(x)\,dx.$



& $c \in [a, b]$

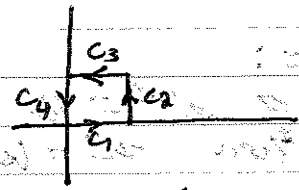
then $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

Suppose C is piecewise smooth, so consists of curves $C_1, C_2, \dots C_n$ that are indiv. smooth & joined end to end.

$$\int_C f(z)\,dz = \sum_{i=1}^{n} \int_{C_i} f(z)\,dz$$

EX:

$$\int_C z e^z \, dz$$



C: the □ w/ vertices $0, 1, i, 1+i$

figure out a parametrization for all $C_1, C_2, C_3, \& C_4$

$$\int_C ze^z dz = \int_{C_1} ze^z dz + \int_{C_2} ze^z dz + \int_{C_3} ze^z dz + \int_{C_4} ze^z dz$$

C_1 : $z = t$, $0 \le t \le 1$

C_2 : $z = 1 + it$, $0 \le t \le 1$

C_3 : $z = t + i$, $1 \le t \le 0 \Rightarrow (1-t) + i$ $0 \le t \le 1$

C_4 : $z = (1-t)i$, $0 \le t \le 1$

Bonus. 10^{pt}

Lingo

A domain is a connected, open, set.

A domain D is simply connected if every simple, closed contour in D & can be shrink to a points w/D and leaving D. (No holes)

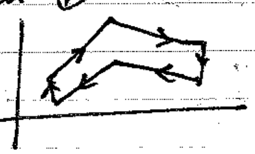
Ex: $\{z \in \mathbb{C} \mid |z| < 1\}$ is simply connected.

If D is not simply connected, then it is multiply connected.

Cauchy's Theorem

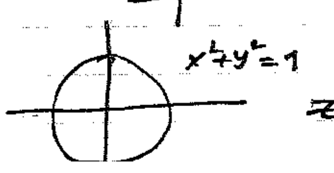
If $f(z)$ - is analytic in a simply connected Domain D, then for every simple closed contour 'C' in D

$$\oint_C f(z)\,dz = 0$$



Ex: $\oint_C \sin z \, dz = 0$

Ex: $\oint_{|z|=1} z^2 dz$



$z = e^{i\theta}$ $\quad 0 \le \theta \le 2\pi$

counterclockwise

$$\int_0^{2\pi} i e^{2i\theta} \cdot e^{i\theta} d\theta = i \int_0^{2\pi} e^{3i\theta} d\theta$$

$$= \frac{i}{3i} e^{3i\theta} \Big|_0^{2\pi}$$

$$= \frac{1}{3} e^{3i(2\pi)} - \frac{1}{3} e^{3i(0)}$$

$$= \frac{1}{3} e^{6\pi i} - \frac{1}{3} e^{0}$$

$$= \frac{1}{3}(e^{6\pi i} - 1) \Longrightarrow e^{6\pi i} \Rightarrow \cos 6\pi + i \sin 6\pi$$

$$1 + 0 = 1$$

$$= \frac{1}{3}(1-1)$$

$$= 0$$

$$\oint_{|z|=1} \left(z^2 + \frac{1}{z-4}\right) dz = 0, \quad \int_{|z|=4} \left(z^2 + \frac{1}{z-4}\right) dz$$

Domain: $\{z \in \mathbb{C} \mid |z| < 2\}$
Simply connected.
Suppose the domain is multiply connected.
Case 1: D is doubly connected.

$$\oint_C f(z)\,dz = 0$$

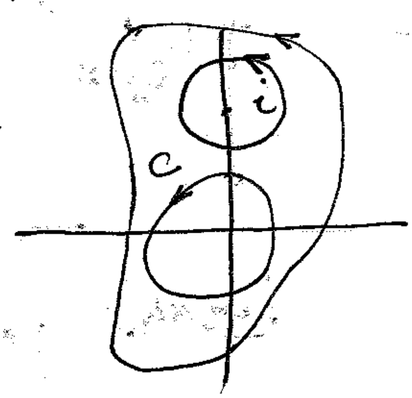
$$0 = \int_{BA} f(z)\,dz + \oint_C f(z)\,dz + \oint_{AB} f(z)\,dz + \oint_{c_1} f(z)\,dz$$

$C \cup AB \cup c_1 \cup BA$ is
closed and doesn't include the hole.

$$-\int_{AB} f(z)\,dz = \int_{BA} f(z)\,dz \quad = 0 \quad \oint_C f(z)\,dz = \oint_{c_1} f(z)\,dz$$

Ex: $\oint_C \frac{dz}{z-i}$

① c: Any closed curve not surrounding $z = i$ ↗ 0
② c: Any closed curve surrounding $z = i$
$\oint 2\pi i$

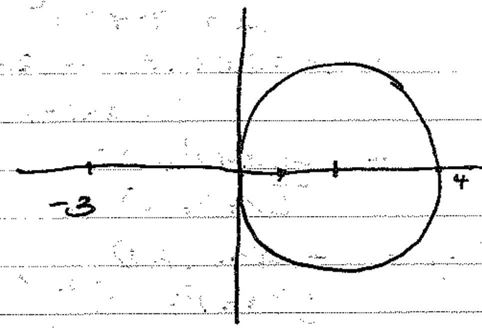


② parametrizing z,
$$z = i + re^{i\theta}, \quad r \ll 0 \quad 0 \le \theta \le 2\pi$$

$$\int_0^{2\pi} \frac{rie^{i\theta}\,d\theta}{i + re^{i\theta} - i} = \int_0^{2\pi} i\,d\theta$$

$$= 2\pi i$$

$\oint_C \frac{5z+7}{z^2+2z-3}\,dz =$ $\qquad |z-2|=2$



$z^2+2z-3=0$

$(z+3)(z-1)=0$

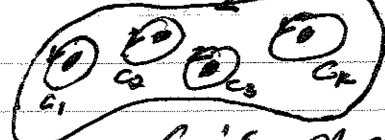
$z=-3,\ z=1$ } not analytic

Rephrase

If $f(z)$ is analytic inside & on a closed contour in a domain D

then $\oint_C f(z)\,dz = 0$

"Deformation of Curve"

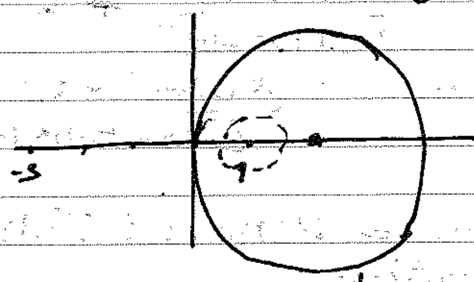


want to compute $\oint_C f(z)\,dz$

Need: $f(z)$ to be analytic inside C but possibly

C_K's shouldn't overlap $\quad \oint_C f(z)\,dz = \sum_{i=1}^{K} \oint_{C_i} f(z)\,dz$

$\oint_C \frac{5z+7}{z^2+2z-3} \to C: |z-2|=2$



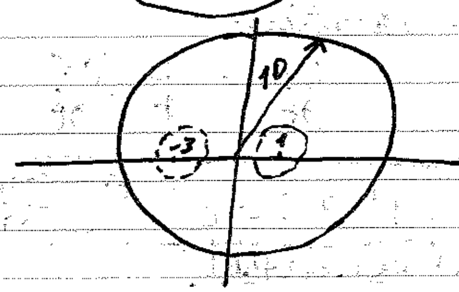
$= \oint_C \left(\frac{2}{z+3} + \frac{3}{z-1}\right) dz$

$= \left[\oint_C \frac{2}{z+3} + \oint_C \frac{3}{z-1}\right] dz$

$= \quad 0 \qquad 3(2\pi i)$

$= \underline{\underline{6\pi i}}$

If C was $|z|=10$



Then $\oint_C \frac{5z+7}{z^2+2z-3} = 4\pi i + 6\pi i$

$= \underline{\underline{10\pi i}}$

(*) $= \oint_C \frac{dz}{(z-a)^n} \quad n>1 \quad$ if C does not contain $z=a$, then $f(z)=0$

if C does contain $z=a$

$z(t) = a + re^{i\theta} \quad 0 \le \theta \le 2\pi$

$dz = ire^{i\theta}\,d\theta$

$\oint_C \frac{dz}{(z-a)^n} = \oint_C \frac{ire^{i\theta}\,d\theta}{(a+re^{i\theta}-a)^n} = \oint_C \frac{ire^{i\theta}}{(re^{i\theta})^n}\,d\theta = i\int (re^{i\theta})^{1-n}$

$= \cancel{\frac{i}{2-n} re^{i\theta(2-n)}} \quad \frac{\cancel{i}\, r^{1-n}}{\cancel{i}(1-n)} e^{i(1-n)\theta}\Big|_0^{2\pi} = \frac{r^{1-n}}{1-n}\left(e^{i(1-n)2\pi} - e^0\right)$

$= \frac{r^{1-n}}{1-n}\left[\cos(1-n)2\pi + i\sin\cancel{(1-n)2\pi}^{\,0}\right] = \frac{r^{1-n}}{1-n}\cdot 0$

$= \cancel{r^{1-n}}\ \ 0 \,//$

$$\oint_C \frac{dz}{(z-a)^n} = \begin{cases} 2\pi i, & n=1 \ \& \ C \text{ enclosed } z=a \\ 0, & \text{all other times} \end{cases}$$

$$\oint_C \frac{dz}{(z+i)^2(z-1)} \qquad C: |z+i| = 2$$

$$\frac{1}{(z+i)^2(z-1)} = \frac{A}{z-1} + \frac{B}{z+i} + \frac{C}{(z+i)^2}$$

$$1 = A(z+i)^2 + B(z-1)(z+i) + C(z-1)$$

$$= Az^2 - 2izA - A + Bz^2 - Bz + Biz - iB + Cz - C$$

$Az^2 + Bz^2 = 0$ $\qquad A + B = 0 \Rightarrow A = -B$

$-2iAz + Biz + Cz = 0$ $\qquad -2iA + Bi + C = 0$ $\qquad 2Bi + Bi + C = 0$ $\qquad \underline{C = -3Bi}$

$-A - iB - C = 1$ $\qquad A + iB + C = -1$

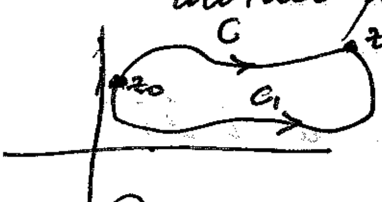
$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ -2i & i & 1 & 0 \\ 1 & i & 1 & -1 \end{array}\right] \xrightarrow{2iR_1+R_2} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 3i & 1 & 0 \\ 1 & i & 1 & -1 \end{bmatrix} \xrightarrow{\frac{R_2}{3}-R_3}$$

Do!

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 3i & 1 & 0 \\ -1 & 0 & -\frac{2}{3} & 1 \end{bmatrix} \xrightarrow{R_1+R_3} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 3i & 1 & 0 \\ 0 & 1 & -\frac{2}{3} & 1 \end{bmatrix} \xrightarrow{R_2/3i} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1/3i & 0 \\ 0 & 1 & -1/3 & 1 \end{bmatrix}$$

Def^n

Let z_0 & z_1 be 2 points in the same complex plane. We say $\oint_C f(z)\,dz$ is independent of path if $\int_C f(z)\,dz$ has the same value for all contours C such that initial point of C is z_0 & the terminal point of C is z_1.



Now Sup. D is simply connected and $f(z)$ is analytic on D.

$$\oint_C f(z)\,dz + \int_{-C_1} f(z)\,dz = 0 \text{ by Cauchy's Theorem}$$

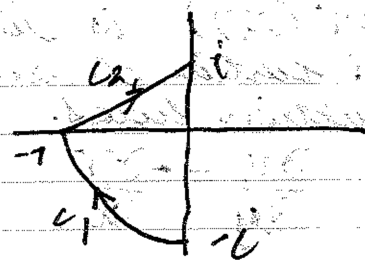
Rewriting $\quad \int_C f(z)\,dz - \int_{C_1} f(z)\,dz = 0$

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz \Rightarrow \int_C f(z)\,dz \text{ is independent of path}$$

Let $f(z)$ be analytic on a simply connected domain D. Then if z_0, z_1 are any pts. in D.

$\int_C f(z)\,dz$ is indep. of path [C: is any path from z_0 to z_1]

EX: $\int_C 3z+1 \, dz$



$C_1: z = e^{i\theta} \quad -\frac{\pi}{2} \le \theta \le -\pi$

$C_2: it - (1-t) \quad 0 \le t \le 1$

line $f(z)$:

$z(t) = z_0 t + z_1(1-t)$

$= it - 1(1-t)$

$0 \le t \le 1$

Another 10^2

Green's Theorem in the Plane.

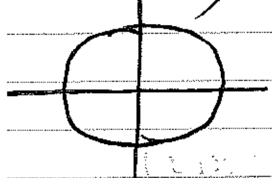
Let C be a closed contour [oriend counter clockwise] in the xy plane. C forms the boundary of a region R in the xy-plane. If $P(x,y)$ & $Q(x,y)$ are continuous w/ continous partial derivative, then

$$\oint P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} + \frac{\partial P}{\partial y} \right) dy\,dx.$$

EX: $\oint -y\,dx + x\,dy$

C: circle radius 2 centered at $(0,0)$

(1) Directely.



$x = 2\cos t$

$y = 2\sin t$

$0 \le t \le 2\pi$

$\int_0^{2\pi} (-2\sin t)(-2\sin t) + 2\cos t(2\cos t)\,dt$

$\int_0^{2\pi} 4\sin^2 t + 4\cos^2 t \,dt = \int_0^{2\pi} 4\,dt = \underline{8\pi}$

(2) Green's Theor.

C: $\iint_R \frac{\partial Q}{\partial x} + \frac{\partial P}{\partial y} \, dy\,dx \Rightarrow \quad Q = x \quad \frac{\partial Q}{\partial x} = 1$

$\quad P = -y \quad \frac{\partial P}{\partial y} = -1$

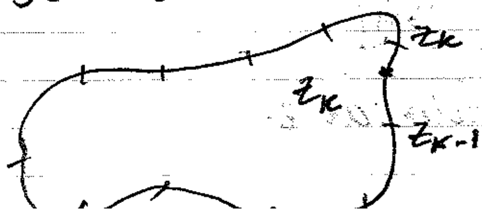
$\iint_R (1 - (-1))\,dy\,dx$

$\iint_R 2\,dy\,dx = \int 2y\,dx = 2xy \oint$ area $= 2 \times 2 \times 2 = 8\pi$

Cauchy's Theorem (Weak version)

$f(z)$ analytic in a simple connected domain D & $f'(z)$ is continuous in D. For every simple closed curve C in D,

$\oint_C f(z)\,dz = 0$



$$\oint_C f(z)\,dz = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(z_k^*)\,\Delta z_k$$

Goal: Any complex line integral can be realized as 2 real line integrals - to which Green's theorem applies.

$$f(z) = u(x,y) + i v(x,y)$$

Let $f(z_k^*) = u_k^*(x,y) + i v_k^*(x,y)$
$= u_k^* + i v_k^*$

$$\Delta z_k = \Delta x_k + i \Delta y_k$$

$$\oint_C f(z)\,dz = \lim_{\|\ \|} \sum (u_k + i v_k)(\Delta x_k + i \Delta y_k)$$

$$= \lim \sum \left[(u_k \Delta x_k - v_k \Delta y_k) + i\,(v_k \Delta x_k + u_k \Delta y_k)\right]$$

$$= \lim_{\|P\|\to 0} \sum_{k=1}^{n} u_k \Delta x_k - v_k \Delta y_k + i \lim_{\|P\|\to 0} v_k \Delta x_k + u_k \Delta y_k$$

$$= \int_C u\,dx - v\,dy + i \int_C v\,dx + u\,dy$$

By Green's theorem

① $\oint_C u\,dx - v\,dy = \iint_R -\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\,dy\,dx = 0$

② $\oint_C v\,dx + u\,dy = \iint_R \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\,dy\,dx = 0$

$f(z)$ is analytic & $f'(z)$ is continuous!

$\Rightarrow$ C-R equation holds;

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Calc I:

$f(x)$ continuous w/ antiderivative $F(x)$.

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

value of the integral is dependent only on the endpoints.

Similarly:

Let $f(z)$ be continuous on a domain D, and suppose $F(z)$ is an antiderivative of $f(z)$ on D, for any contour C in D w/ initial pt z_0 and terminal point z_1.

$$\int f(z)\,dz = F(z_1) - F(z_0)$$

$$\int_C 3z+1\,dz = = \frac{3}{2}z^2+z\Big|_{-i}^{i}$$

$$= \underline{\underline{2i}}$$

proof:

$C: z(t) \quad a \le t \le b$

$z(a) = z_0$

$z(b) = z_1$

$$\int_C f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt$$

$$= \int_a^b F'(z(t))\,z'(t)\,dt$$

$$= \int_a^b \frac{d}{dt}\left[F(z(t))\right]dt$$

$$= F(z(t))\Big|_a^b$$

$$= F(z(b)) - F(z(a))$$

$$= F(z_1) - F(z_0)$$

Corrolary:

If $f(z)$ is continuous on a domain D, and has an antiderivative on D, then $\int_C f(z)\,dz$ is independent of path.

FACT:

If $f(z)$ is independent of path for C points z_0, z_1 in D; then $f(z)$ has an antiderivative.

Dipse: $\int_C \frac{1}{z}\,dz$, $|z|=1 \neq \operatorname{Ln} z\Big|_{|z|=1}$

$\quad \| $

$2\pi i$

Cauchy's Integral Formula

Theorem.

Suppose $f(z)$ is analytic inside and on a simple closed contour C in a simply connected Domain D. Then if a is any point on the interior of C,

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz.$$

EX: $\oint_C \frac{e^z}{z-2}$ $\quad C: |z|=3 \quad f(z) = e^z$

$a = 2 \quad . \quad f(a) = f(2) = e^2$

$$\oint \frac{f(z)}{z-a} = f(a)(2\pi i)$$

$$= e^2(2\pi i)$$

If $C = |z|=1$ the entire function is analytic $\Rightarrow \int = \underline{0}$

EX: $\oint_C \frac{4z}{z-i}\,dz \qquad C: |z|=2$

easy and hard way.

1) $f(z) = 4z \quad , a = i$

$f(i) = \underline{\underline{4i}}$

$$\int \frac{4z}{z-i} = 4i(2\pi i) = \underline{\underline{-8\pi}}$$

2)

$z = 2e^{i\theta}, \quad 0 \le \theta \le 2\pi$

$dz = 2ie^{i\theta}\, d\theta$

$$\int_0^{2\pi} \frac{8e^{i\theta}}{2e^{i\theta}-i} \cdot 2ie^{i\theta} d\theta = 16i \oint_0^{2\pi} \frac{e^{2i\theta}}{2e^{i\theta}-1}\, d\theta$$

let $z = i + e^{i\theta}$

$dz = ie^{i\theta} d\theta$

$$\int_0^{2\pi} \frac{4(i+e^{i\theta})}{\cancel{e^{i\theta}}} i\cancel{e^{i\theta}} d\theta = 4i\int_0^{2\pi} i + e^{i\theta}\, d\theta$$

$$= 4i\left(i\theta + \frac{e^{i\theta}}{i}\right)_0^{2\pi}$$

$$= 4i\left[(i(2\pi) - i^{2\pi i}) - i\right]$$

$$= 4i(2\pi i)$$

$$= \underline{-8\pi}$$

Cauchy's Integral Formula for Derivatives.

Suppose $f(z)$ analytic inside and on a simply closed contour C in a s.c domain D. a is any point interior to C. Then:

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}}\, dz$$

EX: $\oint_C \frac{z+1}{z^4+2iz^3}\, dz \qquad C: |z| = 1$

$$z^4 + 2iz^3 = z^3(z+2i)$$

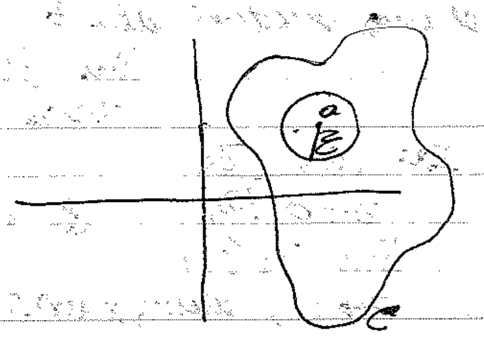
$$\oint_C \frac{\frac{z+1}{(z+2i)}}{z^3}\, dz \qquad f(z) = \frac{z+1}{z+2i}$$

$$a = 0 \quad , n = 2$$

$$\oint_C \frac{z+1}{z^4+2iz^3} = \frac{f^{(2)}(0)(2\pi i)}{2!} = \pi i f''(0) \quad \pi i\left(\frac{\cdot 2(2i-1)}{-8i}\right)$$

$$= \underline{\underline{\frac{\pi(2i-1)}{4}}}$$

Proof of Cauchy's Integral formula



$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a}\, dz.$

proof. lets compute

$$\oint_C \frac{f(z)}{z-a}\, dz. = \oint_{C'} \frac{f(z)}{z-a}\, dz$$

$C': |z-a| = \varepsilon$

$z = a + \varepsilon e^{i\theta}$

$dz = i\varepsilon e^{i\theta}\, d\theta$

$0 \le \theta \le 2\pi$

$$\oint_0^{2\pi} \frac{f(a+\varepsilon e^{i\theta})}{\cancel{a}+\varepsilon e^{i\theta}-\cancel{a}} \cdot i\varepsilon e^{i\theta}\, d\theta$$

$$\oint_0^{2\pi} f(a+\varepsilon e^{i\theta})\, \frac{i\,\cancel{\varepsilon e^{i\theta}}}{\cancel{\varepsilon e^{i\theta}}}\, d\theta$$

$$i\oint_0^{2\pi} f(a+\varepsilon e^{i\theta})\, d\theta$$

we can replace this w/ a circle of arbitrary small radius and not change the answer.

$$\oint_C \frac{f(z)}{z-a}\, dz = \lim_{\varepsilon \to 0} i \int_0^{2\pi} f(a+\varepsilon e^{i\theta})\, d\theta$$

since f is cts.

$$= i \int_0^{2\pi} f\left[\lim_{\varepsilon\to 0} (a+\varepsilon e^{i\theta})\right] d\theta$$

$$= i \int_0^{2\pi} f(a)\, d\theta = i\, f(a)\, \theta \Big|_0^{2\pi}$$

$$= 2\pi i\, f(a)$$

Second proof:

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}}\, dz$$

Example:

$$\oint_C \frac{e^{z^2}}{(z-i)^3}\, dz, \qquad f(z) = e^{z^2} \qquad C: |z-i| = 1$$

$a = i$

$n = 2$

$$\oint_C \frac{e^{z^2}}{(z-i)^3}\, dz = 2\pi i\, f^{(n)}(a)/n!$$

$$= 2\pi i\, f''(i)/2!$$

$$= 2\pi i\,(-2/e)/2 \cdot 1$$

$$= -2\pi i/e$$

$f'(z) = e^{z^2} \cdot 2z = 2z e^{z^2}$

$f''(z) = 2[e^{z^2} + z(e^{z^2}) \cdot 2z]$

$= 2[e^{z^2} + 2z^2 e^{z^2}]$

$f''(i) = 2[e^{i^2} + 2i^2 e^{i^2}]$

$= 2[e^{-1} - 2e^{-1}] = 2[-e^{-1}]$

$= -2/e$

A Sequence is a function whose domain is the set of positive integers and range is contained in the complex #s.

If $\lim z_n = L < \infty$, then $\{z_n\}$ converges to L.

An infinite series of complex numbers has the shape $\sum_{k=1}^{n} a_k$

does $\sum_{n=1}^{\infty} \frac{i}{n(n+1)}$ converge $\quad \frac{a^{n+1}}{a^n}$

$$\sum_{n=2}^{\infty} \frac{i^n}{(1+i)^{n-1}} \to = \sum_{n=1}^{\infty} \frac{i^{n+1}}{(1+i)^n}, \quad \frac{i^{n+2}}{(1+i)^{n+1}} \cdot \frac{(1+i)^n}{i^{n+1}}$$

$$= \frac{i^1}{i}(1+i)^{-1}$$

$$= \frac{i}{(1+i)}$$

$$\frac{i^n(1+i)}{(1+i)^n} = (1+i)\left(\frac{1}{1+i}\right)^n$$

$$\Rightarrow a = (1+i), \quad z = 1/_{1+i}$$

$$\sum az^n = \cancel{\frac{1}{a-1}} \frac{1}{1-a} = \frac{1}{1-(1+i)} = \frac{1}{i} = 1/_i$$