HIGH SCHOOL MATHEMATICS CONTEST
Sponsored by
THE MATHEMATICS DEPARTMENT
of
WESTERN CAROLINA UNIVERSITY

COMPREHENSIVE TEST
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Prepared by:

Geoff Goehle Andrew Chockla Andrew Penland

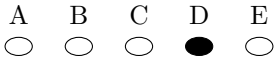
DIRECTIONS:

Do not open this booklet until you are told to do so.

This is a test of your competence in high school mathematics. For each of the 30 problems there are listed up to 5 possible answers. You are to work each problem and determine which is the correct answer. Indicate your choice by making a heavy black mark in the correct place on the separate answer sheet provided. Here is a sample question and answer:

1. If $2x = 3$, then x equals:
   (A) $\frac{2}{3}$ (B) 3 (C) 6 (D) $\frac{3}{2}$ (E) None of the choices (A) through (D) is correct.

The correct answer for the sample is "$\frac{3}{2}$," which is answer (D); therefore, you should answer this question by making a heavy black mark under space D as indicated below.



If you should change your mind about an answer, be sure to erase completely. Do not mark more than one answer for any question. If you are unable to work any particular problem, it is to your advantage to guess at the answer rather than leave it blank. Make no stray marks of any kind on your answer sheet.

When told to do so, open your test booklet to page 2 and begin work. When you have finished one page, go on to the next page. The working time for the entire test is 70 minutes.

The use of calculators is not permitted.

1. A perfect number is a positive integer such that it is equal to the sum of all positive integers smaller than it which divide it. Which of the following is perfect?

   (A) 4 (B) 12 (C) 6 (D) 10 (E) None of the choices (A) through (D) is correct.

2. Emily leaves her home and jogs due east for $5\sqrt{3}$ miles. She then jogs 2 miles in the direction of 30° north of west. How far is Emily from her home?

   (A) 7 miles (B) $\sqrt{79}$ miles (C) 8 miles (D) $\sqrt{71}$ miles
   (E) None of the choices (A) through (D) is correct.

3. If two fair *eight*-sided dice are rolled, what is the probability that their product is divisible by 3?

   (A) 3/64 (B) 7/64 (C) 3/8 (D) 7/16 (E) None of the choices (A) through (D) is correct.

4. The expression $\dfrac{\cos^2(\theta) - \sin^2(\theta)}{1 - \tan^2(\theta)}$ is equal to

   (A) $\dfrac{1}{\cos^2(\theta)}$ (B) $\sin^2(\theta)$ (C) $\dfrac{1}{\sin^2(\theta)}$ (D) $\cos^2(\theta)$
   (E) None of the choices (A) through (D) is correct.

5. A hungry hunter came upon two farmers, one of whom had three loaves of bread, and the other five, all of the same size. The loaves were divided equally among the three and the hunter paid 8 dollars for his share. How should the shepherds divide the money?

   (A) 1 and 7 (B) 2 and 6 (C) 3 and 5 (D) 4 and 4
   (E) None of the choices (A) through (D) is correct.

6. In a deck of 40 cards, each card is labeled with a number between 1 and 10. There are four cards labeled 1, four cards labeled 2, and so on. How many ways are there to draw two cards (at the same time) so that the sum of the numbers on the cards is 10?

   (A) 10 (B) 50 (C) 70 (D) 100 (E) None of the choices (A) through (D) is correct.

7. If $\tan(\theta) = \frac{3}{7}$ and θ lies in the 3rd quadrant, what is $\sin(\theta)$?

   (A) $\frac{3}{\sqrt{58}}$ (B) $\frac{7}{\sqrt{58}}$ (C) $-\frac{3}{\sqrt{58}}$ (D) $-\frac{7}{\sqrt{58}}$ (E) None of the choices (A) through (D) is correct.

8. Evaluate $\dfrac{(1+i)^{104}}{(1-i)^{103}}$, where i is the imaginary number.

   (A) $-1-i$ (B) $1+i$ (C) $-i$ (D) $1-i$ (E) None of the choices (A) through (D) is correct.

9. Which set below is the set of values of b such that the polynomial $2x^2 + bx + 7$ has two real roots?

   (A) $(-\infty, -2\sqrt{14}) \cup (2\sqrt{14}, \infty)$ (B) $(-\sqrt{14}, \sqrt{14})$ (C) $(-\infty, -\sqrt{14}) \cup (\sqrt{14}, \infty)$
   (D) $(-2\sqrt{14}, 2\sqrt{14})$ (E) None of the choices (A) through (D) is correct.

10. A cylindrical tank with radius 4 feet and an open top is partially filled with pudding. A solid cone with radius 2 feet and height 3 feet is suspended vertex down above the pudding so that the vertex just touches the top of the pudding. The cone is lowered at a rate of 3 feet per minute. Assuming that the pudding does not overflow, how long does it take until the cone is completely submerged.

(A) 30 seconds (B) 45 seconds (C) 55 seconds (D) 1 minute
(E) None of the choices (A) through (D) is correct.

11. How many ways are there to color the vertices of a square, using the four colors red, blue, green, and purple if adjacent vertices are required to be different colors?

(A) 24 (B) 60 (C) 84 (D) 108 (E) None of the choices (A) through (D) is correct.

12. If $\cos(\theta) = c$ and θ is an angle in the first quadrant, which expression is equivalent to $\tan^2(\theta)$?

(A) $\frac{c^2}{1+c^2}$ (B) $\frac{c^2}{1-c^2}$ (C) $\frac{1+c^2}{c^2}$ (D) $\frac{1-c^2}{c^2}$ (E) None of the choices (A) through (D) is correct.

13. If $f(x) = \dfrac{x-4}{3}$ and $g(x) = \sqrt[3]{x+8}$, then $f^{-1} \circ g^{-1}(3)$ is

(A) 109 (B) 61 (C) 101 (D) 53 (E) None of the choices (A) through (D) is correct.

14. Suppose $\dfrac{64^{x-y}}{4^{2x}} = 16$ and $\dfrac{3^{2y}}{9^{x-y}} = 81$, find the value of xy

(A) 40 (B) -100 (C) 56 (D) 320 (E) None of the choices (A) through (D) is correct.

15. A rectangular floor with dimensions 24×40 is covered by square tiles of side length 1. A chalk line is drawn from one corner to the diagonally opposite corner. How many tiles have a chalk line segment on them?

(A) 40 (B) 56 (C) 63 (D) 64 (E) None of the choices (A) through (D) is correct.

16. If

$$\begin{bmatrix} x & 6 \\ 2 & y \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} 7x \\ 10 \end{bmatrix}$$

what is the largest possible value of $x + y$?

(A) 4 (B) 9 (C) 15 (D) 50 (E) None of the choices (A) through (D) is correct.

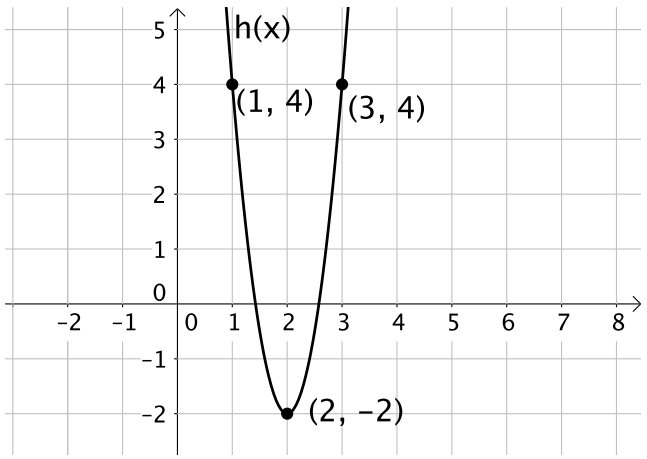
17. Solve $\log_2(x^2 + 2x) = \log_2(5x + 4)$ for x

(A) $x = -1$ (B) $x = -1$ or $x = 4$ (C) $x = 2$ (D) $x = 4$
(E) None of the choices (A) through (D) is correct.

18. If each term of the sequence $\{a_1, a_2, \ldots, a_n\}$ is either 1 or -1 then $a_1 + a_2 + \ldots + a_n$ is always

(A) even if n is odd, odd if n is even (B) odd if n is odd, even if n is even (C) 0 (D) 1
(E) None of the choices (A) through (D) is correct.

19. Suppose $g(x) = 3(x+2)^2$ and the graph of $h(x)$, shown below, displays a transformation of $g(x)$.



Which of the following is the appropriate way to write $h(x)$ in terms of $g(x)$?

(A) $h(x) = \frac{1}{2}g(x-4)-2$ (B) $h(x) = 2g(x+4)+2$ (C) $h(x) = 2g(x-4)-2$ (D) $h(x) = \frac{1}{2}g(x+4)+2$
(E) None of the choices (A) through (D) is correct.

20. Which of the following real valued polynomials contain the imaginary root $-2i$ and another root k?

(A) $x^3 - kx^2 + 5x - 5k$ (B) $x^2 + (-k+2)x - 2k$ (C) $x^3 - kx^2 + 4x - 4k$ (D) $x^2 + (k+2)x + 2k$
(E) None of the choices (A) through (D) is correct.

21. Let x and y be real numbers such that $x + y = 4$ and $x^2y + xy^2 + 3x + 3y = 4$. Find the value of $\dfrac{(xy)^2}{x+y}$

(A) -1 (B) $\dfrac{-1}{2}$ (C) $\dfrac{1}{2}$ (D) -2 (E) None of the choices (A) through (D) is correct.

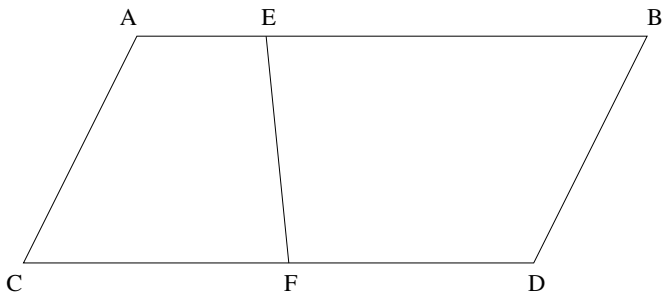
22. Let S be the sum of all multiples of 3 between 1 and 100. What is the value of S?

(A) 1677 (B) 1680 (C) 1683 (D) 1686 (E) None of the choices (A) through (D) is correct.

23. What is the solution to the inequality $|x^2 - 4| \geq x + 2$

(A) $(-\infty, -2] \cup [3, \infty)$ (B) $(-\infty, 1] \cup [3, \infty)$ (C) $(-\infty, -2] \cup [1, \infty)$ (D) $(-\infty, 1) \cup (3, \infty)$
(E) None of the choices (A) through (D) is correct.

24. Consider the following figure where $ABCD$ is a parallelogram.



If $AB = 21$ and $AE = 3$ and the area of $ACFE$ is half the area of $EFDB$ then CF has length

(A) 18 (B) 11 (C) 10 (D) 3 (E) None of the choices (A) through (D) is correct.

25. Let P be the product of any 3 consecutive odd integers. The largest integer dividing all such P is

(A) 2 (B) 3 (C) 7 (D) 15 (E) None of the choices (A) through (D) is correct.

26. If $\ln(ex^2) = a$ and $2a - 2 = b$, where e is the natural number, what is $\dfrac{\ln(x^8)}{b}$?

(A) 1 (B) 2 (C) e (D) 4 (E) None of the choices (A) through (D) is correct.

27. Geraldine has a bag with a certain number of marbles, 4 of which are red and the rest are blue. Each morning Geraldine reaches in the bag and pulls out two marbles, looks at them, and then puts them back. She has found that the probability that she chooses two marbles of the same color is 4/9. How many blue marbles does she have?

(A) 2 (B) 4 (C) 5 (D) 6 (E) 8

28. What is the remainder when $x^{60} - 3x^{31} + 2$ is divided by $x^2 - 1$?

(A) $3x - 3$ (B) $-3x - 3$ (C) 3 (D) $3x + 3$ (E) None of the choices (A) through (D) is correct.

29. If n is a positive integer, what is the largest number of real roots the equation $x^{2n} - x^n = 0$ could have?

(A) 1 (B) 2 (C) 3 (D) $2n$ (E) None of the choices (A) through (D) is correct.

30. Given that

$$\sqrt{1 + 2\sqrt{1 + 2\sqrt{1 + \cdots}}}$$

is a real number, its value is

(A) $\sqrt{2}$ (B) $1 + \sqrt{2}$ (C) $1 - \sqrt{2}$ (D) 1 (E) None of the choices (A) through (D) is correct.