HIGH SCHOOL MATHEMATICS CONTEST
Sponsored by
THE MATHEMATICS DEPARTMENT
of
WESTERN CAROLINA UNIVERSITY

LEVEL III TEST
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Prepared by:

Cory Howk Geoff Goehle Nathan Borchelt

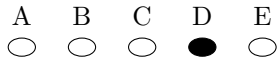
DIRECTIONS:

Do not open this booklet until you are told to do so.

This is a test of your competence in high school mathematics. For each of the 30 problems there are listed up to 5 possible answers. You are to work each problem and determine which is the correct answer. Indicate your choice by making a heavy black mark in the correct place on the separate answer sheet provided. Here is a sample question and answer:

1. If $2x = 3$, then x equals:
   (A) $\frac{2}{3}$ (B) 3 (C) 6 (D) $\frac{3}{2}$ (E) None of the answers (A) through (D) is correct.

The correct answer for the sample is "$\frac{3}{2}$," which is answer (D); therefore, you should answer this question by making a heavy black mark under space D as indicated below.



If you should change your mind about an answer, be sure to erase completely. Do not mark more than one answer for any question. If you are unable to work any particular problem, it is to your advantage to guess at the answer rather than leave it blank. Make no stray marks of any kind on your answer sheet.

When told to do so, open your test booklet to page 2 and begin work. When you have finished one page, go on to the next page. The working time for the entire test is 70 minutes.

The use of calculators is not permitted.

1. While three guards were guarding a jewelry store, a thief slipped in and stole some jewels. On his way out he met the three guards one after another, and to each in turn he gave a half of the jewels he had and then one besides. After he escaped he was left with two jewels. How many had he stolen originally?

   (A) 8 (B) 11 (C) 30 (D) 35 (E) None of the answers (A) through (D) is correct.

2. Consider the circle given by the equation $x^2 - 2x + y^2 + 6y - 39 = 0$. Let (h, k) be the center of this circle and r be the length of its radius. What is the value of $\dfrac{r-k}{h}$?

   (A) 4 (B) 52 (C) 10 (D) 49 (E) None of the answers (A) through (D) is correct.

3. The function $f(x) = x^2 + kx + 9$, for real k, has two roots, r_1 and r_2. What can we say about the product $r_1 r_2$?

   (A) $r_1 r_2$ is always real, and $r_1 r_2 \geq 0$ for all k (B) $r_1 r_2$ is always real, and $r_1 r_2 \leq 0$ for all k
   (C) $r_1 r_2$ is always real, but may be positive or negative, depending on k (D) $r_1 r_2$ is always complex
   (E) $r_1 r_2$ will be complex for some values of k, and real for other values of k

4. Solve for x: $\dfrac{(2+i)x + (1+3i)(-5+i)}{1+2i} = 1$

   (A) $\dfrac{23}{5} - \dfrac{18}{5}i$ (B) $\dfrac{34}{5} + \dfrac{23}{5}i$ (C) $\dfrac{7}{5} - \dfrac{37}{5}i$ (D) $\dfrac{2}{5} + \dfrac{41}{5}i$ (E) None of the answers (A) through (D) is correct.

5. Consider the function $y = -3\sin(\pi x) + 5$. Let A = amplitude and P = period. What is the value of $A + P$?

   (A) 7 (B) 2 (C) 8 (D) 5 (E) None of the answers (A) through (D) is correct.

6. Solve $4^{x+2} = \dfrac{8^{9x-1}}{16^{3x+2}}$.

   (A) 13/15 (B) 4/3 (C) 9/7 (D) 17/5 (E) None of the answers (A) through (D) is correct.

7. If S, T, and V are any sets then $(S \cap T) \cup (S \cap V)$ is the same set as

   (A) S (B) $T \cup V$ (C) $T \cap V$ (D) $S \cap (T \cup V)$ (E) None of the answers (A) through (D) is correct.

8. Consider the equation of a line that passes through the vertex of the parabola $x^2 + 2x + y = 5$ and is perpendicular to the line $y = 2x - 3$. This line can be written in the form $y = mx + b$. Which of the following is equal to $\dfrac{m}{b}$?

   (A) -2/11 (B) -1/11 (C) -4/5 (D) -3/5 (E) None of the answers (A) through (D) is correct.

9. Define $x \diamond y = x^2 + y$, where x, y are complex numbers in standard form $a + bi$. Find the value of $(3 + 2i) \diamond (1 - 3i)$.

(A) $6+9i$ (B) $9-7i$ (C) $4-i$ (D) $9+5i$ (E) None of the answers (A) through (D) is correct.

10. At Galactic Intervention Academy, 50% of students ate a sandwhich at lunch today, 40% had a salad, and 20% had both a sandwhich and a salad. If a student from that school is selected at random, what is the probability that the student had either a sandwhich or a salad today?

(A) 0.4 (B) 0.7 (C) 0.8 (D) 0.9 (E) None of the answers (A) through (D) is correct.

11. Consider the system of equations

$$kx + 2y = a$$
$$2x + ky = b$$

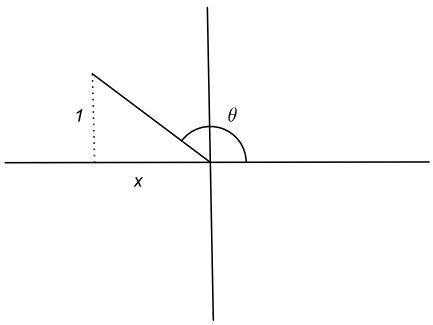
where a, b are real numbers. What is the value of $x + y$?

(A) $\dfrac{k+b}{a+3}$ (B) $\dfrac{a+b}{k+2}$ (C) $\dfrac{3+b}{2+a}$ (D) $\dfrac{k+3}{ab}$ (E) None of the answers (A) through (D) is correct.

12. Find the equation of the quadratic that passes through $(0, 3)$, $(1, 6)$, and $(3, 18)$.

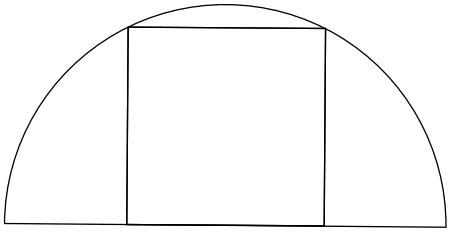
(A) $y = 18x^2 + 6x + 3$ (B) $y = \frac{3}{2}x^2 + \frac{9}{2}x + 3$ (C) $y = x^2 + 2x + 3$ (D) $y = 3x^2 + 6x + 18$ (E) None of the answers (A) through (D) is correct.

13. Given that $\sin(\theta) = \frac{1}{\sqrt{3}}$ and $\frac{\pi}{2} < \theta < \pi$, find the exact value of $\tan(\theta)$.



(A) $\frac{2}{\sqrt{3}}$ (B) $\frac{-2}{\sqrt{3}}$ (C) $\frac{1}{\sqrt{2}}$ (D) $\frac{-1}{\sqrt{2}}$ (E) None of the answers (A) through (D) is correct.

14. A square of area π is inscribed in a semicircle as shown. What is the area of the semicircle?



(A) $\frac{5\pi^2}{8}$ (B) $\frac{5\pi}{4}$ (C) $\frac{3\pi}{8}$ (D) $3\pi^2$ (E) None of the answers (A) through (D) is correct.

15. For how many real values of p is $\sqrt{120-\sqrt{p}}$ an integer?
(A) 5 (B) 6 (C) 10 (D) 11 (E) None of the answers (A) through (D) is correct.

16. The expression $\sqrt[3]{x^3}+\sqrt{x^2}$ is always equal to
(A) x (B) $2x$ (C) 2 (D) 0 (E) None of the answers (A) through (D) is correct.

17. If $f(x)=x^2$ and given $n \neq k$ then
$$\frac{f(f(n))-f(f(k))}{f(n)+f(k)}$$
is equal to
(A) n^2+k^2 (B) n^2-k^2 (C) $\frac{n^2+k^2}{n^2-k^2}$ (D) $\frac{n^2-k^2}{n^2+k^2}$ (E) None of the answers (A) through (D) is correct.

18. Find k such that $x-2$ is a factor of $x^4+x^3-7x^2+kx+6$.
(A) 1 (B) 0 (C) -3 (D) -2 (E) None of the answers (A) through (D) is correct.

19. Three disks labeled one to three are put in a bag. Three other disks labeled one to three are put in a second bag. A disk is drawn from each bag and the two disks are stacked in a pile on a table. This is repeated two more times. What is the probability that at least one of the stacks will contain disks with the same number?
(A) 2/3 (B) 1/2 (C) 1/3 (D) 1/4 (E) None of the answers (A) through (D) is correct.

20. The sum of three consecutive positive integers is always

(A) even (B) odd (C) a prime number (D) divisible by 3 (E) None of the answers (A) through (D) is correct.

21. Find the sum of the solutions to $9^x - 12 \cdot 3^x + 27 = 0$.

(A) 2 (B) 8 (C) 3 (D) 12 (E) None of the answers (A) through (D) is correct.

22. A point A is chosen outside a circle with center C. A tangent from A meets the circle at B, while the segment AC meets the circle at P. Given $AB = 8$ and $AP = 2$, the radius of the circle must equal

(A) $2\sqrt{8}$ (B) 15 (C) 30 (D) $2\sqrt{17}$ (E) None of the answers (A) through (D) is correct.

23. Recall that all consecutive elements of an arithmetic sequence satisfy $a_{n+1} = a_n + d$ for some difference constant d. Suppose that we have two arithmetic sequences: $\{a_n\}$ defined by difference constant d_a, and $\{b_n\}$ defined by difference constant d_b. Suppose that

$$\frac{a_4}{a_1} = 7, \qquad \frac{b_4}{b_2} = 5, \qquad \frac{d_a}{d_b} = 4.$$

If $a_1 = 1$, then what is b_1?

(A) -1/2 (B) 1/8 (C) -1/4 (D) 3/8 (E) None of the answers (A) through (D) is correct.

24. Town A and Town B are 45 miles apart. Car A leaves Town A at noon, and drives toward Town B at 40 mph. At 1pm, Car B leaves Town B at 44 mph, driving along the same road toward Town A. At 12:45pm, Car A breaks down, so the driver starts riding his bike toward Town B at 16mph. At what time do the two drivers meet?

(A) 1:08pm (B) 1:09pm (C) 1:10pm (D) 1:11pm (E) 1:12pm

25. Find the solution of the inequality: $\Big|2x + |1 - 3x| - 4\Big| \geq 2$

(A) $(-\infty, -4] \cup [-2, \frac{2}{5}] \cup [\frac{7}{5}, \infty)$ (B) $(-\infty, -5] \cup [-1, \frac{3}{5}] \cup [\frac{7}{5}, \infty)$ (C) $(-\infty, -3] \cup [-1, \frac{4}{5}] \cup [\frac{8}{5}, \infty)$
(D) $(-\infty, -2] \cup [-1, \frac{2}{5}] \cup [\frac{4}{5}, \infty)$ (E) None of the answers (A) through (D) is correct.

26. Suppose that we define the functions

$$f(x) = \sqrt{x - 5}$$
$$g(x) = 3x + 1$$
$$h(x) = f\Big(g^{-1}(\frac{1}{x})\Big).$$

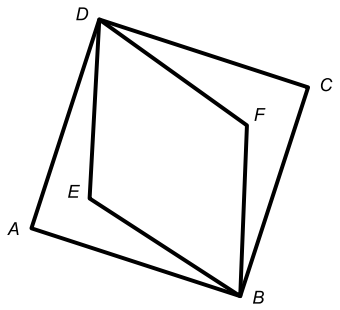
What is the domain of $h(x)$?

(A) $x \leq \frac{1}{16}, x \neq 0$ (B) $x \leq 5, x \neq 0$ (C) $x \leq \frac{-1}{3}$ (D) $x \leq \frac{4}{3}, x \neq 0$ (E) None of the answers (A) through (D) is correct.

27. If $|x - \ln(y)| = x + \ln(y)$ where x and $\ln(y)$ are real then

(A) $x = 0$ (B) $y = 1$ (C) $x = 0$ or $y = 1$ (D) $x = 0$ and $y = 1$ (E) None of the answers (A) through (D) is correct.

28. Rhombus $ABCD$ is similar to rhombus $BFDE$. The area of rhombus $ABCD$ is 24, and $\angle BAD = 60°$. What is the area of rhombus $BFDE$?



(A) 6 (B) 8 (C) $4\sqrt{3}$ (D) 12 (E) None of the answers (A) through (D) is correct.

29. Solve for x: $\log_3(2x + 1) + \log_{27}(81) = \log_9(5)$

(A) $\dfrac{5^{1/2}3^{-4/3} - 1}{2}$ (B) $\dfrac{5^{-3/2}3^{3/4} - 1}{2}$ (C) $\dfrac{5^{2/3}3^{-1/2} - 1}{2}$ (D) $\dfrac{5^{4/3}3^{-2/3} - 1}{2}$ (E) None of the answers (A) through (D) is correct.

30. What is the sum of all 4 digit numbers that can be formed using the 4 distinct digits 1, 2, 3 and 4?

(A) 29616 (B) 57600 (C) 6666 (D) 159984 (E) None of the answers (A) through (D) is correct.