HIGH SCHOOL MATHEMATICS CONTEST
Sponsored by
THE MATHEMATICS DEPARTMENT
of
WESTERN CAROLINA UNIVERSITY

COMPREHENSIVE TEST
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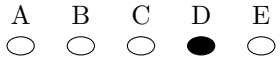
DIRECTIONS:

Do not open this booklet until you are told to do so.

This is a test of your competence in high school mathematics. For each of the 30 problems there are listed up to 5 possible answers. You are to work each problem and determine which is the correct answer. Indicate your choice by making a heavy black mark in the correct place on the separate answer sheet provided. Here is a sample question and answer:

1. If $2x = 3$, then x equals:
   (A) $\frac{2}{3}$ (B) 3 (C) 6 (D) $\frac{3}{2}$ (E) None of the answers (A) through (D) is correct.

The correct answer for the sample is "$\frac{3}{2}$," which is answer (D); therefore, you should answer this question by making a heavy black mark under space D as indicated below.



If you should change your mind about an answer, be sure to erase completely. Do not mark more than one answer for any question. If you are unable to work any particular problem, it is to your advantage to guess at the answer rather than leave it blank. Make no stray marks of any kind on your answer sheet.

When told to do so, open your test booklet to page 2 and begin work. When you have finished one page, go on to the next page. The working time for the entire test is 70 minutes.

The use of calculators is not permitted.

1. If $-1 < x < 0$, then $|x - \frac{1}{x}| + |\frac{1}{x} + x|$ is equal to
   (A) $-2x$ (B) $2x$ (C) 0 (D) $-\frac{2}{x}$ (E) $\frac{2}{x}$

2. A palindrome, such as 64546, is a number that remains the same when its digits are reversed. The numbers x and $x + 32$ are three-digit and four-digit palindromes, respectively. What is the sum of the digits of x?
   (A) 21 (B) 22 (C) 23 (D) 24 (E) None of the answers (A) through (D) is correct.

3. Which of the following tables represent two events, A and B, that are independent?

   I

	A	$\sim$ A
B	17	41
$\sim$ B	16	26

   II

	A	$\sim$ A
B	15	45
$\sim$ B	10	30

   III

	A	$\sim$ A
B	20	37
$\sim$ B	33	10

   (A) I only (B) II only (C) III only (D) Both I and III
   (E) None of the answers (A) through (D) is correct.

4. For what value of k will the system of equations
$$\begin{aligned} 2x - 3y &= -4 \\ -3x + ky &= 6 \end{aligned}$$
   have an infinite number of solutions?
   (A) 2 (B) 9/2 (C) 4/3 (D) 1 (E) None of the answers (A) through (D) is correct.

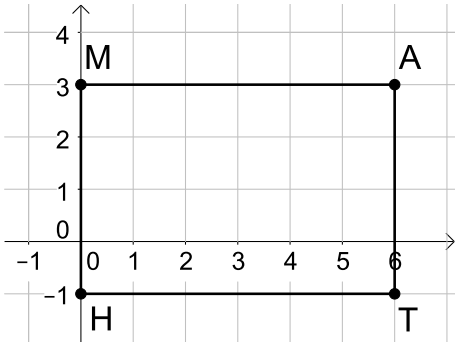
5. Suppose n is a positive integer and that $\ln(x)$ denotes the natural log function. Find n if
$$\ln(2 + 4 + 6 + 8 + \ldots + 2n) = \ln(n) + 2\ln(10) - \ln(2)$$
   (A) 10 (B) 49 (C) 101 (D) $\ln(29)$ (E) $\ln(101)$

6. Suppose that $f(x) = \dfrac{5x + 2}{x - 3}$ and $g(x) = x^7 + 3x + 4$. Both $f(x)$ and $g(x)$ are invertible. Calculate $(g \circ f)^{-1}(4)$.
   (A) $-2/5$ (B) $-1/3$ (C) 4/7 (D) 5/4 (E) None of the answers (A) through (D) is correct.

7. The sum of all the digits appearing in the first fifty positive integers is
   (A) 225 (B) 330 (C) 1275 (D) 150 (E) None of the answers (A) through (D) is correct.

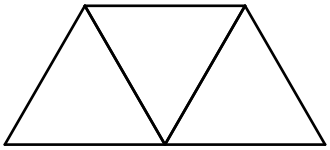
8. Andrew is traveling on the edges of triangle RST. Starting from vertex R, he randomly moves to another vertex every five seconds. What is the probability, that after 4 moves, he is at vertex R?
   (A) 3/8 (B) 5/8 (C) 7/16 (D) 9/16 (E) None of the answers (A) through (D) is correct.

9. The rectangle shown below has vertices M at (0,3), A at (6,3), T at (6,-1), and H at (0,-1). If the center of dilation and rotation is the origin, dilate the rectangle using a scale factor of $\frac{1}{2}$, then rotate the resulting figure counterclockwise by 90°. What is the sum of the y-coordinates of vertices A and T after applying the transformation described above?



(A) 2 (B) 3 (C) -1 (D) 1 (E) None of the answers (A) through (D) is correct.

10. Suppose 3 equilateral triangles are placed as shown in the image below



If the outer perimeter of the object shown above is 40 feet, what is the area of the object?

(A) $16\sqrt{3}$ ft^2 (B) $32\sqrt{5}$ ft^2 (C) $48\sqrt{3}$ ft^2 (D) $96\sqrt{5}$ ft^2
(E) None of the answers (A) through (D) is correct.

11. One of Thom and Tom always lies on Tuesdays, Wednesdays and Thursdays, and always tells the truth on the other days of the week. The other always lies on Fridays, Saturdays and Sundays, and always tells the truth the other days of the week. At noon, the two had the following conversation:

Thom: I lie on Sundays.

Tom: I will lie tomorrow.

Thom: I lie on Mondays.

This conversation takes place on a

(A) Monday (B) Tuesday (C) Wednesday (D) Thursday (E) Friday

12. If $f(x) = x^4 + x - 1$ and n is a non-zero positive integer, which of the following must be true about $f(n)$?

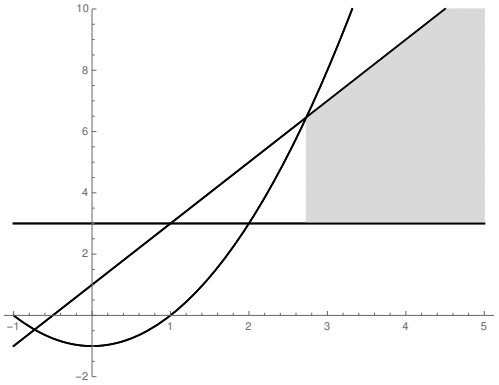
(A) $f(n) > n$ (B) $f(n)$ is an odd integer (C) $f(n)$ is non-zero
(D) More than one of the choices (A) through (C) is correct
(E) None of the choices (A) through (D) is correct.

13. Suppose $f(x) = x^a$ where a is a positive real number. If $f(f(f(2))) = 64$ then a satisfies
(A) $0 < a < 1$ (B) $a = 1$ (C) $1 < a < 2$ (D) $a = 2$ (E) $a > 2$

14. The angle between the hour hand and the minute hand of a clock at 2:40 PM is equal to
(A) 120° (B) 140° (C) 160° (D) 180° (E) None of the answers (A) through (D) is correct.

15. The product of three prime numbers is 47 times the sum of the numbers. One of the numbers must be
(A) 7 (B) 17 (C) 37 (D) 47 (E) None of the answers (A) through (D) is correct.

16. Which of the following inequalities has its solution represented by the shaded region in the graph below?



(A) $3 \le x^2 - 1 \le y \le 2x + 1$ (B) $x^2 - 1 \le 2x + 1 \le y \le 3$ (C) $3 \le 2x + 1 \le y \le x^2 - 1$
(D) $3 \le y \le 2x + 1 \le x^2 - 1$ (E) None of the answers (A) through (D) is correct.

17. The equation $8^{4x+3} = 16^{2-x^2}$ has two solutions, r_1 and r_2. What is their sum, $r_1 + r_2$?
(A) -4 (B) -3 (C) -2 (D) -1 (E) None of the answers (A) through (D) is correct.

18. Find the exact value of $\sin(2\cot^{-1}\frac{1}{3})$.
(A) $\frac{3\sqrt{2}}{2}$ (B) $\frac{3\sqrt{10}}{10}$ (C) $\frac{3}{5}$ (D) $\frac{3\sqrt{10}}{5}$ (E) None of the answers (A) through (D) is correct.

19. Halfway through a 100-shot archery tournament, Symone leads by 50 points. For each shot a bullseye scores 10 points, with other possible scores being 8, 4, 2, and 0. Symone always scores at least 4 points on each shot. If n of Symone's remaining shots are bullseyes she will be guaranteed victory. What is the minimum value for n?
(A) 38 (B) 40 (C) 42 (D) 44 (E) None of the answers (A) through (D) is correct.

20. Consider the quadrilateral ABCD where AB=21, BC=3, CD=15, AD=15 and $\angle$ADC forms a right angle. What is the value of $\angle$ABC?

(A) 30° (B) 45° (C) 60° (D) 90° (E) None of the answers (A) through (D) is correct.

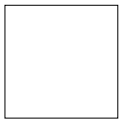
21. A farmer has a compost pile in a cylindrical bin whose diameter is 30 inches. Each week the farmer dumps food scraps from the house into the compost pile and promptly covers the food scraps with a layer of horse manure 3 inches thick (3 inches high). The farmer collects the horse manure in a cylindrical garbage can whose diameter is 20 inches. If the farmer can fill the garbage can 9 inches deep in 10 minutes, how many minutes will it take to have enough horse manure to make a layer 3 inches deep in the compost pile?

(A) 6 minutes (B) 6.5 minutes (C) 7 minutes (D) 7.5 minutes (E) 8 minutes

22. A certain university has a logo that is a rotating cube in which one face has a W on it and the other five faces are blank. Originally the W-face is at the front of the cube as shown in (A) below. Then the following sequence of moves are repeated over and over:

1. rotate the cube 90° around a horizontal axis, so that the front face moves counter clockwise;
2. rotate the cube 90° around a horizontal axis, so that the front face moves down;
3. rotate the cube 90° around the vertical axis, so that the front face moves to the left;
4. rotate the cube 90° around the same horizontal axis, and in the same direction, as step 2.

After this sequence has been repeated a total of 2017 times the front face will look like

(A)  (B)  (C)  (D)  (E) 

23. Let s be the sum of all rational numbers which are roots of the polynomial

$$x^4 - x^3 - 2x^2 + 6x - 4.$$

Which of the following gives the correct description of s?

(A) s is a rational number, but it is not an integer (B) s is a positive integer (C) s is equal to zero (D) s is a negative integer (E) There is not enough information given to determine s.

24. When $x^{57} - 3x^{21} + 6$ is divided by $2x + 2$ the remainder is

(A) 2 (B) 4 (C) 6 (D) 8 (E) None of the answers (A) through (D) is correct.

25. If $a + b = \frac{1}{2}$ and $a^2 + b^2 = 1$, then what does $a^3 + b^3$ equal?

(A) 7 (B) $\frac{11}{16}$ (C) $\frac{-3}{8}$ (D) $\frac{1}{8}$ (E) None of the answers (A) through (D) is correct.

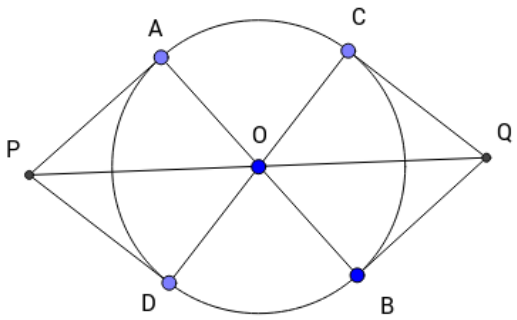
26. Consider a rectangle ABCD with a point, P, inside the rectangle. Assuming AP=4, BP=6, and CP=5, find the value of DP.

(A) $\sqrt{5}$ (B) $\sqrt{7}$ (C) $\sqrt{11}$ (D) $\sqrt{13}$ (E) None of the answers (A) through (D) is correct.

27. Rod and Todd play a game where each person alternates drawing one marble from an urn, which contains 5 unmarked marbles and 1 marble marked "Win". The marbles are otherwise indistinguishable, and the marble that is selected is replaced and remixed in the urn after each draw. The person that draws the "Win" marble wins the game, and play continues until someone wins. If Todd gets to draw first, what is the probability that Todd wins the game?

(A) 1/4 (B) 5/11 (C) 1/2 (D) 6/11 (E) 3/4

28. Consider the figure shown below.



Suppose $\overline{AP}$, $\overline{DP}$, $\overline{CQ}$, and $\overline{BQ}$ are all tangents to the circle which has center O. If $AB = 2$ and $\angle AOC = 60°$ then PQ is equal to

(A) 4 (B) 3 (C) 2 (D) 1 (E) None of the answers (A) through (D) is correct.

29. Suppose Cup A and Cup B both have water in them. A student walked into the room and decided to pour water between the two cups. First, she poured 2/3 of the water from Cup B into Cup A. She then poured 1/4 of the water from Cup A into Cup B. Finally, she poured 1/2 of the water from Cup B into Cup A. After the student performed these steps, Cup A had 26 fluid ounces of water and Cup B had 6 fluid ounces of water. How much water did Cup A contain before the student came into the room?

(A) 4 (B) 12 (C) 16 (D) 28 (E) None of the choices (A) through (D) is correct.

30. Let $f(x) = 5\sin(x)$. How many times does the graph of $y = f(x)$ cross the x-axis for x-values in the interval $[1, 315]$?

(A) 80 (B) 100 (C) 150 (D) 200 (E) None of the other answer choices is correct.