HIGH SCHOOL MATHEMATICS CONTEST
Sponsored by
THE MATHEMATICS DEPARTMENT
of
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LEVEL III TEST
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DIRECTIONS:

Do not open this booklet until you are told to do so.

This is a test of your competence in high school mathematics. For each of the 30 problems there are listed up to 5 possible answers. You are to work each problem and determine which is the correct answer. Indicate your choice by making a heavy black mark in the correct place on the separate answer sheet provided. Here is a sample question and answer:

1. If $2x = 3$, then x equals:

   (A) $\frac{2}{3}$ (B) 3   (C) 6   (D) $\frac{3}{2}$ (E) None of the answers (A) through (D) is correct.

The correct answer for the sample is "$\frac{3}{2}$," which is answer (D); therefore, you should answer this question by making a heavy black mark under space D as indicated below.

If you should change your mind about an answer, be sure to erase completely. Do not mark more than one answer for any question. If you are unable to work any particular problem, it is to your advantage to guess at the answer rather than leave it blank. Make no stray marks of any kind on your answer sheet.

When told to do so, open your test booklet to page 2 and begin work. When you have finished one page, go on to the next page. The working time for the entire test is 70 minutes.

The use of calculators is not permitted.

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1. The expression
$$\frac{\log_2 64}{\log_2 \sqrt{32}}$$

   is equal to

   (A) $\log_2(64 - \sqrt{32})$ (B) $\log_{32} 64$ (C) 1   (D) 12/5
   (E) None of the answers (A) through (D) is correct.

2. The number of five-digit numbers which remain unchanged when their digits are written in reverse order
   is

   (A) 500   (B) 900   (C) 1000   (D) 5000   (E) None of the answers (A) through (D) is correct.

3. If the circle given by the equation $x^2 - 4x + y^2 - 8y - 44 = 0$ has a radius of r, find r.

   (A) 8   (B) 16   (C) 32   (D) 64   (E) None of the choices (A) through (D) is correct.

4. Suppose you have a bag which contains discs marked 1, 2, 3, 4 and 5. You draw two discs from the bag,
   without replacement. What is the probability that their sum is odd?

   (A) 1/2   (B) 3/5   (C) 2/3   (D) 3/4   (E) None of the answers (A) through (D) is correct.

5. The sum of five numbers is 100. The sum of the first two numbers is 26, the sum of the second and third
   number is 48, the sum of the third and fourth number is 57, and the sum of the last two numbers is 32.
   The third number is

   (A) 20   (B) 42   (C) 15   (D) 9   (E) None of the answers (A) through (D) is correct.

6. For which value of b does the function $f(x) = -x^2 + bx - 75$ have a maximum value of 25?

   (A) -50 (B) 50   (C) -20 (D) 30   (E) None of the answers (A) through (D) is correct.

7. A cylinder has a height of 2 and a non-zero radius of r. If the cylinder has the same volume as a sphere
   with radius r, what is the value of r?

   (A) $\dfrac{\sqrt{2}}{8}$ (B) $\dfrac{3}{4\pi}$ (C) $\dfrac{3}{2}$ (D) $\dfrac{3}{8}$ (E) None of the choices (A) through (D) is correct.

8. Which of the following equations have *exactly* the same graph:

   $$I : y = (x-1)(x+2)^2 \qquad II : y = x^3 + 3x - 4 \qquad III : y = \frac{(x-1)^2(x+2)^2(x-3)}{(x-1)(x-3)}$$

   (A) *I* and *II* only   (B) *I* and *III* only   (C) *II* and *III* only   (D) *I, II,* and *III*
   (E) None of the answers (A) through (D) is correct.

9. Let i be such that $i^2 = -1$. Then

$$\frac{1}{1 + \frac{1}{1 + \frac{1}{i}}}$$

is equal to

(A) $\frac{3}{5} - \frac{i}{5}$ (B) $2 - i$ (C) i (D) $\frac{1}{2} + \frac{2}{3}i$ (E) None of the answers (A) through (D) is correct.

10. Which of the following describes the solution x of the equation

$$3^{x+2} = \frac{81^{x-2}}{27(9^{x+1})}$$

(A) An odd integer   (B) An even integer   (C) A positive rational number which is not an integer
(D) A positive real number which is not a rational number
(E) None of the choices (A) through (D) is correct.

11. If θ is an angle in the second quadrant with $\cos(\theta) = -\frac{2}{7}$, what is $\csc(\theta)$?

(A) $\frac{3\sqrt{5}}{7}$ (B) $-\frac{3\sqrt{5}}{7}$ (C) $\frac{7\sqrt{5}}{15}$ (D) $-\frac{7\sqrt{5}}{15}$
(E) None of the choices (A) through (D) is correct.

12. Suppose b and c are real numbers. If $x^2 + bx + c$ has $2 + 3i$ as a root, what is $\frac{b}{c}$?

(A) $\frac{-2}{3}$ (B) $\frac{2}{3}$ (C) $\frac{-4}{13}$ (D) $\frac{4}{13}$ (E) None of the choices (A) through (D) is correct.

13. Suppose $f(x) = 3x + 1$ and $g(x) = 5x - 2$. Which set gives the solutions of $|f(g(x)) - f(x)| \geq 2$.

(A) $\left(\frac{2}{3}, \infty\right)$ (B) $\left(\frac{1}{3}, \frac{2}{3}\right)$ (C) $\left(-\infty, \frac{-1}{3}\right) \cup \left(\frac{2}{3}, \infty\right)$ (D) $\left(-\infty, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \infty\right)$
(E) None of the choices (A) through (D) is correct.

14. If $f(x) = x^4 + x - 1$ and n is a non-zero positive integer, which of the following must be true about $f(n)$?

(A) $f(n) \geq n$ (B) $f(n)$ is an odd integer   (C) $f(n)$ is non-zero
(D) More than one of the choices (A) through (C) is correct
(E) None of the choices (A) through (D) is correct.

15. Suppose Cup A and Cup B both have water in them. A student walked into the room and decided to pour water between the two cups. First, she poured 2/3 of the water from Cup B into Cup A. She then poured 1/4 of the water from Cup A into Cup B. After the student performed these steps, Cup A had 21 liters of water and Cup B had 11 liters of water. How much water did Cup A contain before the student came into the room?

(A) 4   (B) 12   (C) 20   (D) 28   (E) None of the choices (A) through (D) is correct.

16. One solution of the equation
$$0 = x^3 - 8x^2 + 16x - 3$$
is $x = 3$. Find the sum of the remaining solutions.

(A) 5   (B) $\dfrac{5}{2}$ (C) $\dfrac{5 + 2\sqrt{21}}{2}$ (D) 8   (E) None of the answers (A) through (D) is correct.
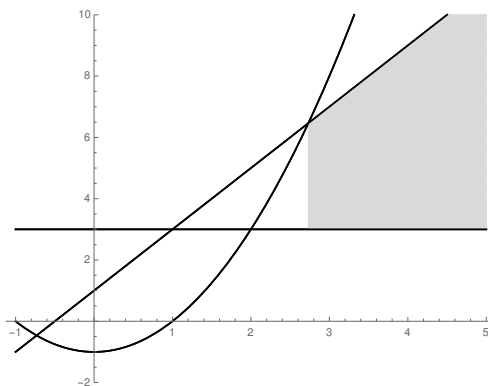
17. Solve the following equation:
$$9^x - 3^{x+1} = 10$$

(A) $x = 5$ (B) $x = -2$ (C) $x = \log_3 -2$ (D) $x = \log_3 5$
(E) None of the answers (A) through (D) is correct.

18. Suppose $f(x) = x^a$ where a is a positive real number. If $f(f(f(2))) = 64$ then a satisfies
(A) $0 < a < 1$ (B) $a = 1$ (C) $1 < a < 2$ (D) $a = 2$ (E) $a > 2$

19. Which of the following inequalities has its solution represented by the shaded region in the graph below?

(A) $3 \le x^2 - 1 \le y \le 2x + 1$ (B) $x^2 - 1 \le 2x + 1 \le y \le 3$ (C) $3 \le 2x + 1 \le y \le x^2 - 1$
(D) $3 \le y \le 2x + 1 \le x^2 - 1$ (E) None of the answers (A) through (D) is correct.

20. The sum of all the digits appearing in the first fifty positive integers is
(A) 225   (B) 330   (C) 1275   (D) 150   (E) None of the answers (A) through (D) is correct.

21. One store sells AA batteries for \$2 per battery and AAA batteries for \$1 per battery. A second store sells AA batteries for \$3 per battery and AAA batteries for \$4 per battery. You buy n AA batteries and m AAA batteries from each store and you spend \$30 overall. How many batteries total did you buy?
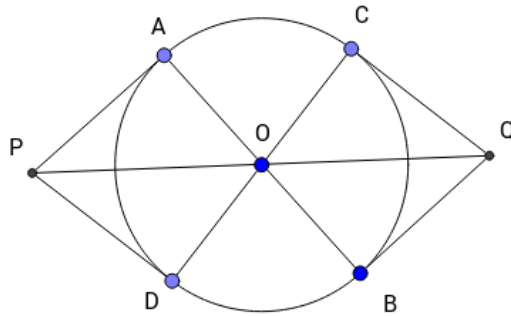(A) 6   (B) 10   (C) 12   (D) 24   (E) None of the answers (A) through (D) is correct.

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22. The roots of the equation $(\sin\theta + \cos\theta)^2 = 0$ are

(A) all multiples of π (B) all odd multiples of π (C) all odd multiples of $\pi/2$
(D) all odd multiples of $\pi/4$ (E) None of the answers (A) through (D) is correct.

23. Each of ten users in an internet chat room claims that exactly two of the other nine are 30 years old or older. Not all of them are lying. The number of them telling the truth is

(A) 3   (B) 5   (C) 6   (D) 8   (E) Not enough information to determine.

24. For what values of a and b is the equation

$$\left(a^{\ln b}\right)^{ab} = \left(b^{\ln a}\right)^{ba},$$

where ln denotes the natural log, well defined and true?

(A) no values   (B) all non-zero positive values   (C) all non-negative values   (D) all values
(E) None of the answers (A) through (D) is correct.

25. A linear function of the form $f(x) = 3x + b$ has an inverse function of the form $f^{-1}(x) = ax + k$. Which of the following is equal to the product ak?

(A) $\dfrac{-b}{3}$ (B) $\dfrac{b}{3}$ (C) $\dfrac{-b}{9}$ (D) $\dfrac{b}{9}$ (E) None of the choices (A) through (D) is correct.

26. A certain rectangle has vertices $(1,3)$, $(2,4)$, $(6,0)$ and (a,b). What is the value of $a + b$?

(A) 0   (B) 4   (C) 5   (D) 6   (E) None of the choices (A) through (D) is correct.

27. We do not know the formula for $f(x)$, but we know that it satisfies $f(2x + 3) = f(x) + 5$ for any real number x. What is the slope of the line that intersects the graph of $y = f(x)$ at $x = 1$ and at $x = 13$?

(A) $\dfrac{5}{2}$ (B) $\dfrac{5}{3}$ (C) $\dfrac{5}{6}$ (D) $\dfrac{5}{13}$ (E) None of the choices (A) through (D) is correct.
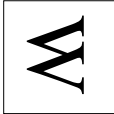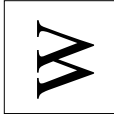
28. Consider the figure shown below.

Suppose $\overline{AP}$, $\overline{DP}$, $\overline{CQ}$, and $\overline{BQ}$ are all tangents to the circle which has center O. If $AB = 2$ and $\angle AOC = 60°$ then PQ is equal to

(A) 4   (B) 3   (C) 2   (D) 1   (E) None of the answers (A) through (D) is correct.

29. A certain university has a logo that is a rotating cube in which one face has a W on it and the other five faces are blank. Originally the W-face is at the front of the cube as shown in (A) below. Then the following sequence of moves are repeated over and over:

1. rotate the cube 90° around a horizontal axis, so that the front face moves counter clockwise;

2. rotate the cube 90° around a horizontal axis, so that the front face moves down;

3. rotate the cube 90° around the vertical axis, so that the front face moves to the left;

4. rotate the cube 90° around the same horizontal axis, and in the same direction, as step 2.

After this sequence has been repeated a total of 2017 times the front face will look like

(A) W   (B) ⩘   (C) Ⱳ   (D) ⩗   (E) ☐

30. When $x^{57} - 3x^{21} + 6$ is divided by $2x + 2$ the remainder is

(A) 2   (B) 4   (C) 6   (D) 8   (E) None of the answers (A) through (D) is correct.