

HIGH SCHOOL MATHEMATICS CONTEST
Sponsored by
THE MATHEMATICS DEPARTMENT
of
WESTERN CAROLINA UNIVERSITY

COMPREHENSIVE TEST
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Prepared by:

Nathan Borchelt
Geoff Goehle

Andrew Chockla
Cory Howk

Axelle Faughn
Andrew Penland

DIRECTIONS:

Do not open this booklet until you are told to do so.

This is a test of your competence in high school mathematics. For each of the 30 problems there are listed up to 5 possible answers. You are to work each problem and determine which is the correct answer. Indicate your choice by making a heavy black mark in the correct place on the separate answer sheet provided. Here is a sample question and answer:

1. If $2x = 3$, then x equals:

- (A) $\frac{2}{3}$ (B) 3 (C) 6 (D) $\frac{3}{2}$ (E) None of the answers (A) through (D) is correct.

The correct answer for the sample is " $\frac{3}{2}$," which is answer (D); therefore, you should answer this question by making a heavy black mark under space D as indicated below.

A B C D E

If you should change your mind about an answer, be sure to erase completely. Do not mark more than one answer for any question. If you are unable to work any particular problem, it is to your advantage to guess at the answer rather than leave it blank. Make no stray marks of any kind on your answer sheet.

When told to do so, open your test booklet to page 2 and begin work. When you have finished one page, go on to the next page. The working time for the entire test is 70 minutes.

The use of calculators is not permitted.

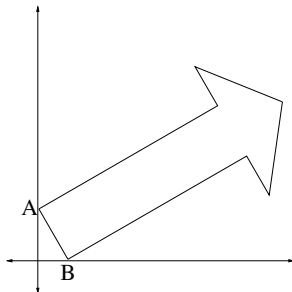
1. Find the sum of the three smallest positive radian solutions to the equation $5 \cos(x) - 2 \cos^2(x) = 2$.
 (A) $\frac{7\pi}{3}$ (B) 2π (C) $\frac{10\pi}{3}$ (D) $\frac{13\pi}{3}$ (E) None of the choices (A) through (D) is correct.
2. Which of the following is equivalent to the statement: $2 \log(x) + 3 \log(x - 3) - 3 \log(2x - 5)$
 (A) $\log\left(\frac{6x^2 - 18}{6x - 15}\right)$ (B) $\log\left(\frac{(2x - 5)^3}{x^2(x - 3)^3}\right)$ (C) $\log\left(\frac{6x - 15}{6x^2 - 18}\right)$ (D) $\log\left(\frac{x^2(x - 3)^3}{(2x - 5)^3}\right)$
 (E) None of the answers (A) through (D) is correct.
3. A group of players are choosing roles for a 6 person team in a popular multiplayer game. They need to have 3 players in the Offense role, 2 players in the Defense role, and 1 player in the Support role. How many different ways are there to distribute the roles among the players?
 (A) 6 (B) 24 (C) 60 (D) 720 (E) None of the answers (A) through (D) is correct.

4. Which of the following is equivalent to

$$\frac{(1 + i)(3 - 2i)}{(2 + i)^2}$$

- (A) $\frac{8}{5} - \frac{3}{5}i$ (B) $\frac{5}{3} + \frac{1}{4}i$ (C) $\frac{11}{25} + \frac{23}{25}i$ (D) $\frac{19}{25} - \frac{17}{25}i$
 (E) None of the answers (A) through (D) is correct.

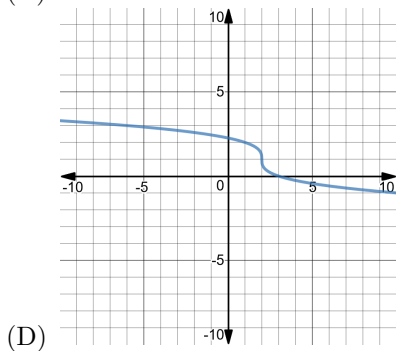
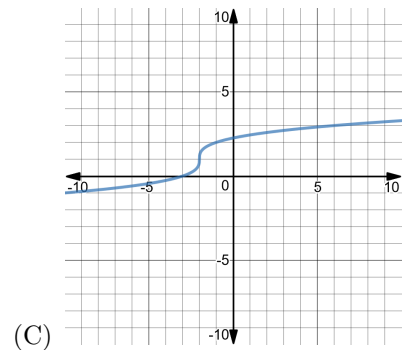
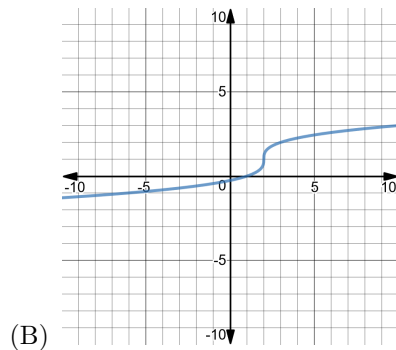
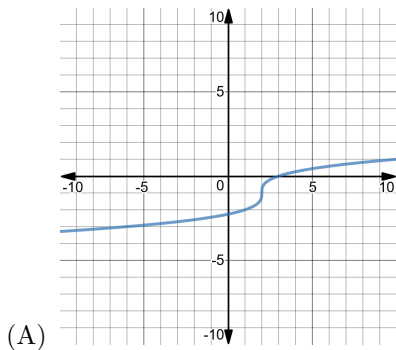
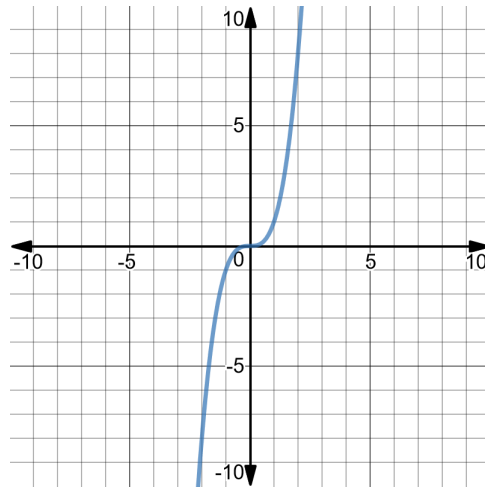
5. Consider the figure shown below



If point A is at $(0, 6)$ and point B is at $(2, 0)$ then the axis of symmetry of the figure is

- (A) $y = \frac{1}{2}x + 2$ (B) $y = \frac{1}{3}x + 2$ (C) $y = \frac{1}{2}x + \frac{7}{3}$ (D) $y = \frac{1}{3}x + \frac{8}{3}$
 (E) None of the answers (A) through (D) is correct.
6. Suppose $f(x) = \frac{6}{x+1}$ and $g(x) = \frac{3}{x}$. What is the value of $f^{-1} \circ g(x)$?
 (A) $18x - 1$ (B) $\frac{2}{x} - 1$ (C) $\frac{18}{x} + 1$ (D) $2x - 1$
 (E) None of the answers (A) through (D) is correct.

7. The following graph represents a function $f(x)$. Which graph represents $f^{-1}(x - 2) + 1$?



(E) None of the answers (A) through (D) is correct.

8. Suppose a bag contains a collection of wooden blocks. The blocks are either triangles or circles, and are colored either red or blue. The distribution of colors and shapes is shown in the table below.

	Red	Blue
Triangle	7	4
Circle	5	9

Two blocks are randomly chosen from the bag, without replacement. If both blocks are red, what is the probability that one block is a triangle and the other block is a circle?

- (A) $\frac{7}{60}$ (B) $\frac{11}{50}$ (C) $\frac{35}{66}$ (D) $\frac{35}{132}$ (E) None of the answers (A) through (D) is correct.

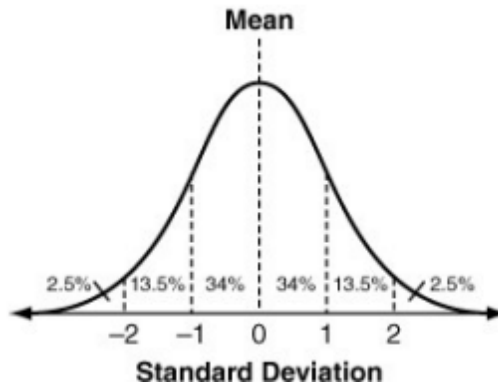
9. Which of the following is equivalent to $\cos(\sin^{-1}(3x))$?

- (A) $\sqrt{1-9x^2}$ (B) $\sqrt{9-x^2}$ (C) $\sqrt{9x^2-1}$ (D) $\frac{1}{\sqrt{9-x^2}}$
(E) None of the answers (A) through (D) is correct.

10. Which of the following describes the graph of the equation $(x+y)^2 = x^2 + y^2$?

- (A) two lines (B) a single point (C) the empty set (D) a circle
(E) None of the answers (A) through (D) is correct.

11. In a college math class, 200 students took a final exam. The final exam results showed students had an average score of 65.3% with a standard deviation of 5.2%. The scores on the final exam followed a normal distribution curve with population percentages as shown below. Approximately how many students scored above 54.9% but below 70.5%?



- (A) 82 (B) 94 (C) 163 (D) 190 (E) None of the answers (A) through (D) is correct.

12. Solve the logarithmic equation $\log_4(x) + \log_2(x) = \frac{-9}{2}$.

- (A) $1/8$ (B) $3/16$ (C) $1/4$ (D) $5/2$ (E) None of the answers (A) through (D) is correct.

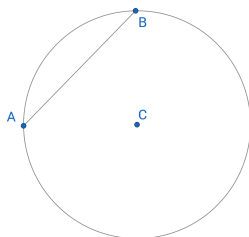
13. Which of the following is the solution to the inequality $|2x + 5| \geq x + |x + 2|$?

- (A) $(-\infty, \frac{-2}{3}] \cup [\frac{2}{3}, \infty)$ (B) all real x (C) $(-\infty, -1] \cup [\frac{-1}{2}, \frac{1}{2}] \cup [2, \infty)$ (D) $[\frac{-7}{2}, \infty)$
(E) None of the choices (A) through (D) is correct.

14. Find the sum of the solutions to $9^x - 30 \cdot 3^x + 100 = 0$.

- (A) 30 (B) $\log_3(15 + 5\sqrt{5})$ (C) $\log_3(100)$ (D) $3 + \log_3(10)$
(E) None of the answers (A) through (D) is correct.

15. Point A is one-fourth of the way around a circle from Point B . If the diameter of the circle is 4, which number is within 0.2 of the length of the chord \overline{AB} ?



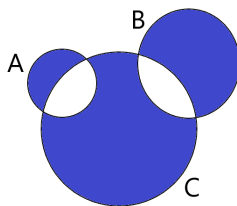
- (A) 2.3 (B) 2.9 (C) 3.2 (D) 3.8 (E) None of the answers (A) through (D) is correct.

16. Suppose that two circles A and B (with radius 5 and 12, respectively) intersect the larger circle C of radius 13 as shown below. The areas of intersection are removed. If we denote

$$x = \text{shaded area of } A + \text{shaded area of } B$$

$$y = \text{shaded area of } C$$

what can we say about the relationship between x and y ?



- (A) $y > 2x$ (B) $x > y$ (C) $x = y$ (D) Not enough information to determine the relationship.
 (E) None of the choices (A) through (D) is correct.

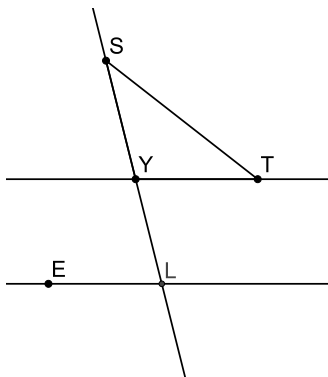
17. Mary, Frank, Susan and June have built houses in a popular video game. The four houses are arranged in a straight line from left to right and are each made from either wood, cobblestone, glass, or dirt. Each house is made from a different material and the following facts are known:

- The third house in the row is cobblestone.
- Mary's house is made from dirt.
- Frank's house is not at either end, but is further right of the glass house.
- Susan owns the fourth house.
- The first house is not made from dirt.

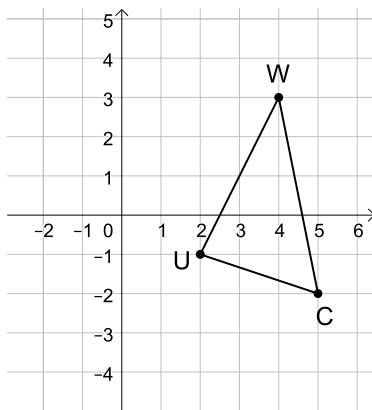
What is June's house made of?

- (A) Wood (B) Cobblestone (C) Glass (D) Dirt
 (E) Not enough information to determine an answer.

18. In the figure shown below, lines \overleftrightarrow{YT} and \overleftrightarrow{EL} are parallel. Suppose triangle STY is an isosceles triangle such that $SY = YT$. If $\angle YTS$ is 37° what is the size of $\angle ELY$?

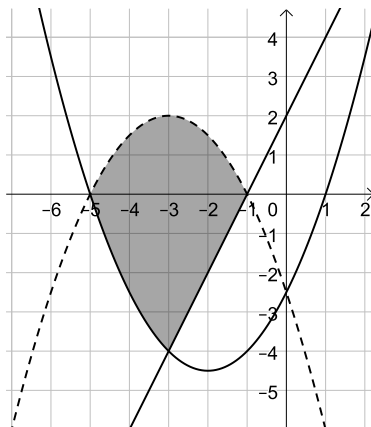


- (A) 74° (B) 106° (C) 34° (D) 146° (E) None of the answers (A) through (D) is correct.
19. Suppose a positive integer n is divisible by 3. Which of the following cannot be equal to n^2 ?
- (A) 1,275,989,841 (B) 7,967,704,644 (C) 2,996,248,644 (D) 146,748,996 (E) 7,978,955,624
20. The triangle shown below has vertices W at $(4,3)$, C at $(5,-2)$, and U at $(2,-1)$. If the center of transformation is the origin, dilate the triangle using a scale factor of $\frac{1}{2}$, then rotate the resulting figure counterclockwise by 270° . What is the sum of the y -coordinates of vertices W and C ?



- (A) 4.5 (B) 1 (C) -4.5 (D) -0.5 (E) None of the answers (A) through (D) is correct.
21. Suppose that $\frac{8x+1}{x^2-13x+42} = \frac{A}{x+B} + \frac{C}{x+D}$. What is $A+B+C+D$?
- (A) -3 (B) 7 (C) 9 (D) -5 (E) None of the answers (A) through (D) is correct.

22. The first four terms of an arithmetic sequence are p , 9, $3p - q$, and $3p + q$. What is the 2018th term of this sequence?
 (A) 8073 (B) 8071 (C) 7975 (D) 7977 (E) None of the answers (A) through (D) is correct.
23. Suppose $n \geq 3$. Which of the following is equivalent to $\frac{n! - (n-1)!}{(n-2)!}$?
 (A) $\frac{n}{n-2}$ (B) $(n-1)^2$ (C) $\frac{n!}{2}$ (D) $(n-3)!$
 (E) None of the choices (A) through (D) is correct.
24. Suppose x is a real number such that $x^2 - 9x + 14 = 2$. Then $3x^3 - 26x^2 + 33x + 10$ is equal to
 (A) 3 (B) 7 (C) $6x - 2$ (D) $6x + 2$ (E) None of the answers (A) through (D) is correct.
25. Which inequalities, listed below the graph, are true about the system of equations that would represent the graph shown below?



I: $y \leq -0.5(x + 3)^2 + 2$

II: $y \geq 0.5(x + 2)^2 - 4.5$

III: $y > 2x + 2$

- (A) Only I is true (B) Only II is true (C) Both I and III are true (D) I, II, and III are true
 (E) None of the answers (A) through (D) is correct.

26. Each cell in a 4x4 array either contains a M, S or K. The eight 4 letter strings formed by the rows, read left to right, and columns, read top to bottom, of the array are, in no particular order:

MSKK MMSK MKSM MKMS
SKMK SKSM KSMK KSKS

The string formed by the top-left to bottom-right diagonal is

- (A) MKMK (B) MSMK (C) MSSK (D) MKSK (E) MMSK

27. The function $f(n)$ is defined on the set of integers and satisfies $f(n) = \begin{cases} n - 3, & \text{if } n \geq 1000 \\ f(f(n + 5)), & \text{if } n < 1000. \end{cases}$

Find the value of $f(84)$.

- (A) 994 (B) 997 (C) 998 (D) 1003 (E) None of the answers (A) through (D) is correct.

28. Simplify: $\sin^6(x) + 3\sin^2(x)\cos^2(x) + \cos^6(x)$

- (A) 1 (B) $\sin(x) + 1$ (C) $\sin(x) + \cos(x)$ (D) $1 + \frac{3}{4}\sin^2(2x)$
(E) None of the answers (A) through (D) is correct.

29. Suppose we define $e^{\theta i} = \cos(\theta) + i \cdot \sin(\theta)$ where $i^2 = -1$. Evaluate

$$\left(e^{\pi+i} e^{-4\pi+2i} \right)^{\frac{1}{2}i}$$

- (A) $e^{3/2}$ (B) $-e^{-3/2}$ (C) $ie^{-3/2}$ (D) $-ie^{3/2}$
(E) None of the choices (A) through (D) is correct.

30. Consider a standard 8×8 checkerboard. The squares alternate colors between red and black, with no adjacent squares having the same color. The squares are numbered in order from 1 through 64. The first square is black, so that all odd numbers are black and all even numbers are red. We place 1 penny on square 1, 2 nickels on square 2, 4 pennies on square 3, 8 nickels on square 4, etc... so that (I) the squares are alternating between pennies and nickels, and (II) each square has twice as many coins as the previous square. Given a penny is worth 1 cent and a nickel is worth 5 cents, what is the monetary value (in cents) of the coins on the checkerboard?

- (A) $\frac{8}{3}(4^{32} - 1)$ cents (B) $\frac{6}{3}(4^{31} - 1)$ cents (C) $\frac{20}{3}(4^{31} - 1)$ cents (D) $\frac{11}{3}(4^{32} - 1)$ cents
(E) None of the choices (A) through (D) is correct.