HIGH SCHOOL MATHEMATICS CONTEST
Sponsored by
THE MATHEMATICS DEPARTMENT
of
WESTERN CAROLINA UNIVERSITY

LEVEL III TEST
March 15, 2018

Prepared by:

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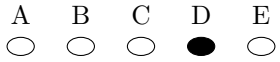
DIRECTIONS:

Do not open this booklet until you are told to do so.

This is a test of your competence in high school mathematics. For each of the 30 problems there are listed up to 5 possible answers. You are to work each problem and determine which is the correct answer. Indicate your choice by making a heavy black mark in the correct place on the separate answer sheet provided. Here is a sample question and answer:

1. If $2x = 3$, then x equals:
   (A) $\frac{2}{3}$ (B) 3 (C) 6 (D) $\frac{3}{2}$ (E) None of the answers (A) through (D) is correct.

The correct answer for the sample is "$\frac{3}{2}$," which is answer (D); therefore, you should answer this question by making a heavy black mark under space D as indicated below.



If you should change your mind about an answer, be sure to erase completely. Do not mark more than one answer for any question. If you are unable to work any particular problem, it is to your advantage to guess at the answer rather than leave it blank. Make no stray marks of any kind on your answer sheet.

When told to do so, open your test booklet to page 2 and begin work. When you have finished one page, go on to the next page. The working time for the entire test is 70 minutes.

The use of calculators is not permitted.

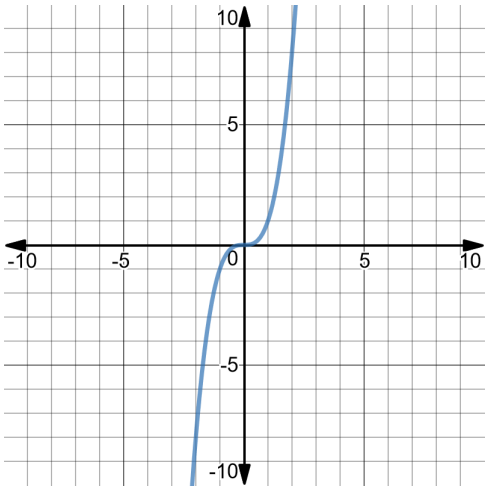
1. Find an equation of the line containing the centers of the two circles $x^2 + y^2 - 2x - 4y - 4 = 0$ and $x^2 + y^2 + 4x + 2y - 10 = 0$.

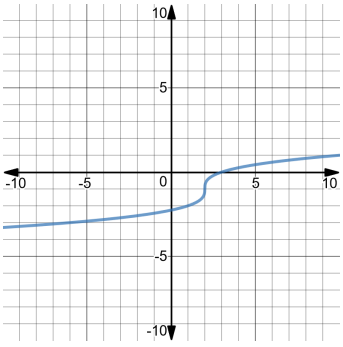
   (A) $-x - y = 1$ (B) $-x + y = -1$ (C) $x + y = -1$ (D) $x - y = -1$
   (E) None of the answers (A) through (D) is correct.

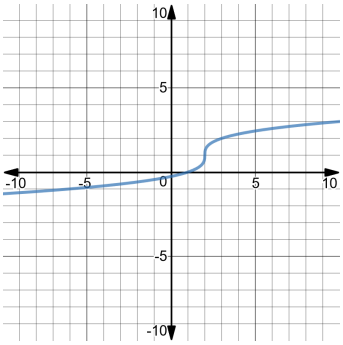
2. Alex has 12 coins in his pocket (a combination of nickels and dimes), worth a combined \$0.80. Jack has 15 coins (another combination of nickels and dimes), worth a combined \$1.15. If Tom has twice as many nickels as Alex, and half as many dimes as Jack, how much money is in his pocket?

   (A) \$1.20 (B) \$1.05 (C) \$1.35 (D) \$1.10 (E) None of the answers (A) through (D) is correct.

3. The following graph represents a function $f(x)$. Which graph represents $f^{-1}(x-2)+1$?



(A)


(B)


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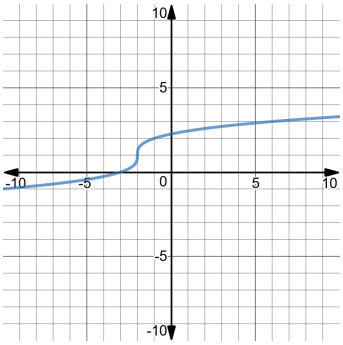
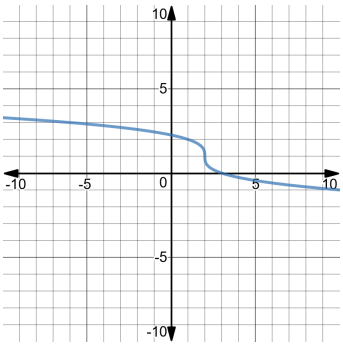


(D)


(E) None of the answers (A) through (D) is correct.

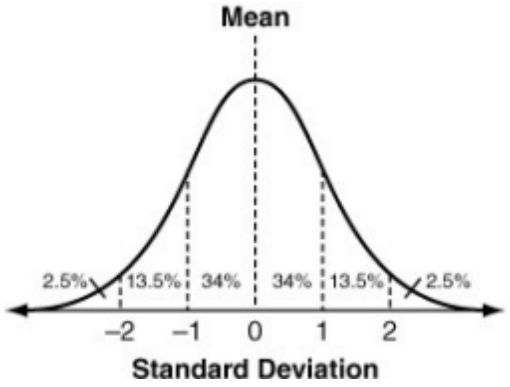
4. Solve the exponential equation $9^{3x+1} = 27^{x-4}$.
   (A) 8/9 (B) -14/3 (C) -2/27 (D) 11/3 (E) None of the choices (A) through (D) is correct.

5. The ratio of a to b is 4:3. The sum of a and b is 14. What is the ratio of $a - b$ to $a + b$?
   (A) 1:6 (B) 7:1 (C) 1:7 (D) 6:7 (E) None of the answers (A) through (D) is correct.

6. Consider the function $h(x) = -3\sin(2x) - 1$. Let P be the period of this function (in radians), A be the amplitude of this function, and $y = K$ represent the equation for the midline of the function. Determine the value of $P + A + K$.
   (A) $\frac{4}{\pi}$ (B) $-4 + \pi$ (C) 6 (D) $1 + \pi$ (E) None of the choices (A) through (D) is correct.

7. A circle with an area of 17π has a sector with a central angle of 54 degrees. What is the exact area of the sector?
   (A) $\frac{27\pi}{17}$ (B) $\frac{17\pi}{7}$ (C) $\frac{34\pi}{13}$ (D) $\frac{13\pi}{20}$ (E) None of the answers (A) through (D) is correct.

8. In a college math class, 200 students took a final exam. The final exam results showed students had an average score of 65.3% with a standard deviation of 5.2%. The scores on the final exam followed a normal distribution curve with population percentages as shown below. Approximately how many students scored above 54.9% but below 70.5%?



   (A) 82 (B) 94 (C) 163 (D) 190 (E) None of the answers (A) through (D) is correct.

9. Which of the following describes the graph of the equation $(x + y)^2 = x^2 + y^2$?
   (A) two lines (B) a single point (C) the empty set (D) a circle
   (E) None of the answers (A) through (D) is correct.

10. Given $f(2x) = \log_2(x)$, find an expression equivalent to $f(x)$.
    (A) $2\log_2(x)$ (B) $\frac{1}{2}\log_2(x)$ (C) $\frac{1}{2} + 2\log_2(x)$
    (D) $-1 + \log_2(x)$ (E) None of the answers (A) through (D) is correct.

11. If $(a+bi)^2 = 7+24i$, where a and b are real numbers, find the value of $|a|+|b|$.

(A) 5 (B) 7 (C) 9 (D) 11 (E) None of the choices (A) through (D) is correct.

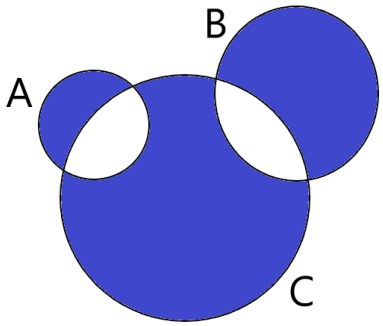
12. The function $f(x) = \dfrac{x^2-12x+11}{x^2+2x-35}$ has a horizontal asymptote at $y=A$, vertical asymptotes at $x=B$ and $x=C$, and x-intercepts at $x=D$ and $x=E$. Determine the value of $\dfrac{B\cdot C - A}{D+E}$.

(A) 3 (B) 6 (C) -6 (D) 1 (E) None of the answers (A) through (D) is correct.

13. Suppose that two circles A and B (radius 5 and 12, respectively) intersect the larger circle C of radius 13 as shown below. The areas of intersection are removed. If we denote

$$x = \text{ shaded area of } A \ + \ \text{shaded area of } B$$
$$y = \text{ shaded area of } C$$

what can we say about the relationship between x and y?



(A) $y > 2x$ (B) $x > y$ (C) $x = y$ (D) Not enough information to determine the relationship.
(E) None of the choices (A) through (D) is correct.

14. Suppose that $\dfrac{8x+1}{x^2-13x+42} = \dfrac{A}{x+B} + \dfrac{C}{x+D}$. What is $A+B+C+D$?

(A) -3 (B) 7 (C) 9 (D) -5 (E) None of the answers (A) through (D) is correct.

15. Which of the following is the solution to the inequality $|2x+5| \geq x+|x+2|$?

(A) $(-\infty, \frac{-2}{3}] \cup [\frac{2}{3}, \infty)$ (B) all real x (C) $(-\infty,-1] \cup [\frac{-1}{2}, \frac{1}{2}] \cup [2,\infty)$ (D) $[\frac{-7}{2}, \infty)$
(E) None of the choices (A) through (D) is correct.

16. Two populations are undergoing continuous exponential growth. Population A grew from 100 to 1000 in two months. Population B has a continuous growth rate that is twice that of A. If population B starts with 100, what will the size of population B be after two months?

(A) 1,500 (B) 2,000 (C) 10,000 (D) 15,000
(E) None of the choices (A) through (D) is correct.

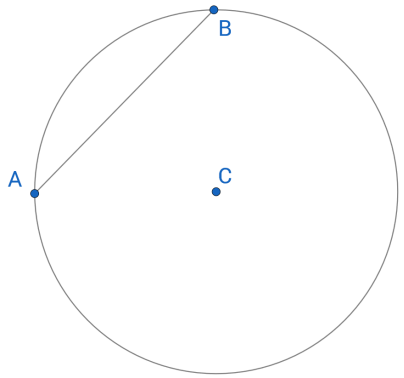
17. Solve the logarithmic equation $\log_4(x) + \log_2(x) = \frac{-9}{2}$.

(A) 1/8 (B) 3/16 (C) 1/4 (D) 5/2 (E) None of the answers (A) through (D) is correct.

18. One solution of the equation $x^3 + 5x^2 + 5x - 2 = 0$ is -2. Find the sum of the remaining solutions.

(A) 0 (B) 3 (C) -3 (D) -2 (E) None of the answers (A) through (D) is correct.

19. Point A is one-fourth of the way around a circle from Point B. If the diameter of the circle is 4, which number is within 0.2 of the length of the chord $\overline{AB}$?

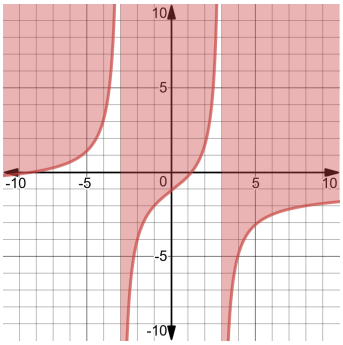


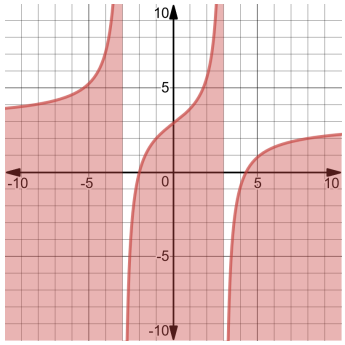
(A) 2.3 (B) 2.9 (C) 3.2 (D) 3.8 (E) None of the answers (A) through (D) is correct.

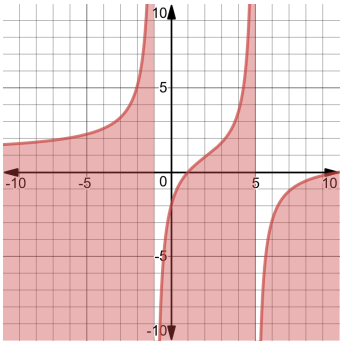
20. If $\theta = \sin^{-1}(x)$, what is $\cos(\theta)$?

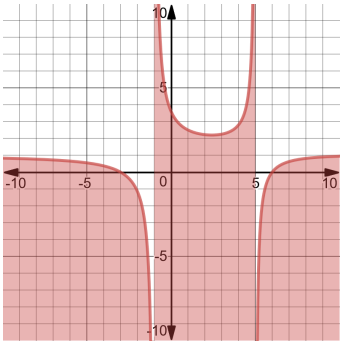
(A) $\sqrt{1-x^2}$ (B) $\sqrt{x^2-1}$ (C) $1+x$ (D) x^2-1
(E) None of the answers (A) through (D) is correct.

21. Which shaded region illustrates the solution of the inequality $y - 2 \leq \dfrac{x^2 - 7x - 8}{x^2 - 9}$?

(A) 

(B) 

(C) 

(D) 

(E) None of the answers (A) through (D) is correct.

22. Find the sum of the three smallest positive radian solutions to the equation $5\cos(x) - 2\cos^2(x) = 2$.

(A) $\frac{7\pi}{3}$ (B) 2π (C) $\frac{10\pi}{3}$ (D) $\frac{13\pi}{3}$ (E) None of the choices (A) through (D) is correct.

23. Suppose you have a bag which contains discs marked 1, 2, 3, 4 and 5. You draw two discs from the bag, without replacement. What is the probability that their product is odd?

(A) 0.3 (B) 0.2 (C) 0.4 (D) 0.24 (E) None of the answers (A) through (D) is correct.

24. A ball of putty is rolled into a perfect sphere, and then sliced into equal hemispheres, each of which has surface area measuring A square units, and volume measuring A cubic units. Find the radius of the original ball.

(A) 2 units (B) 3 units (C) 4 units (D) 5 units
(E) None of the answers (A) through (D) is correct.

25. Define $f(x) = 3x - 4$ and $g(x) = x^5 - 4x^3 + 1$. What is $3f^{-1}(1) + 2g^{-1}(1)$?

(A) 2 (B) 3 (C) 4 (D) 5 (E) None of the answers (A) through (D) is correct.

26. The cities of Robbinsville and Hickory are 160 miles apart. Car A leaves Robbinsville at 12:00pm driving toward Hickory at 45 mph. Car B leaves Hickory at 12:30pm driving along the same road toward Robbinsville at 60 mph. Both cars stop for lunch at 1:00 pm, with Car A back on the road at 1:15pm and Car B back on the road at 1:35pm, again driving 45 mph and 60 mph respectively. At what time do they meet?

(A) 2:00pm (B) 2:05pm (C) 2:10pm (D) 2:15pm
(E) None of the choices (A) through (D) is correct.

27. Find the sum of the solutions to $9^x - 30 \cdot 3^x + 100 = 0$.

(A) 30 (B) $\log_3(15 + 5\sqrt{5})$ (C) $\log_3(100)$ (D) $3 + \log_3(10)$
(E) None of the answers (A) through (D) is correct.

28. The first four terms of an arithmetic sequence are p, 9, $3p - q$, and $3p + q$. What is the 2010th term of this sequence?

(A) 8041 (B) 8039 (C) 7943 (D) 7945 (E) None of the answers (A) through (D) is correct.

29. Suppose $n \geq 3$. Which of the following is equivalent to $\dfrac{n! - (n-1)!}{(n-2)!}$?

(A) $\frac{n}{n-2}$ (B) $(n-1)^2$ (C) $\frac{n!}{2}$ (D) $(n-3)!$ (E) None of the choices (A) through (D) is correct.

30. The function f is defined on the set of integers and satisfies $f(n) = \begin{cases} n - 3, & \text{if } n \geq 1000 \\ f\big(f(n+5)\big), & \text{if } n < 1000. \end{cases}$

Find the value of $f(84)$.

(A) 994 (B) 997 (C) 998 (D) 1003 (E) None of the answers (A) through (D) is correct.